Hamilton Cycles in Plane Triangulations

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Abstract

We extend Whitney's Theorem that every plane triangulation without separating triangles is hamiltonian by allowing some separating triangles. More precisely we define a decomposition of a plane triangulation G into 4-connected 'pieces' and show that if each piece shares a triangle with at most three other pieces then G is hamiltonian. We provide an example to show that our hypothesis that 'each piece shares a triangle with at most three other pieces' cannot be weakened to 'four other pieces'. As part of our proof we also obtain new results on Tutte cycles through specified vertices in planar graphs.

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1 Introduction

We consider finite simple graphs. Let G be a graph, X a set of vertices and edges of G. Then $G - X$ denotes the graph obtained from G by deleting X and all edges incident to some vertex in X. When $X = \{x\}$, we write $G - x$ instead of $G - \{x\}$. For a subgraph H of G, we use $H \cap X$ to denote the set of elements of X which are vertices or edges of H. For convenience, we use $A := B$ to define A as B or to re-name B as A. Let x, y be two vertices in G. Then $G + xy := G$ if $xy \in E(G)$, otherwise $G + xy$ denotes the graph obtained from G by adding the edge xy. A *circuit graph* is an ordered pair (G, C) consisting of a 2-connected plane graph G and a facial cycle C of G such that for any 2-cut U of G, each component of $G - U$ contains a vertex of C. A *plane chain of blocks* is a plane graph with blocks $B_1, B_2, \ldots, B_k$ such that for $i, j \in \{1, 2, \ldots, k\}$, $B_i \cap B_j$ consists of exactly one vertex if $|i - j| = 1$, $B_i \cap B_j = \emptyset$ for $|i - j| \geq 2$, and $(\bigcup_{i=1}^{k} B_i) - B_j$ is in the infinite face of B_j. We also say that B_1 and B_k are the *endblocks*. Given a path P in a plane graph G we define the *plane chain of blocks in G along P* to be the union of all blocks of G which contain an edge of P. (It can easily be seen that this union is indeed a plane chain of blocks.) For a cycle C in a plane graph and $x, y \in V(C)$, $xCy := x$ if $x = y$, and otherwise, xCy denotes the subpath of C from x to y in the clockwise direction. The *outerwalk* D in a plane graph is the facial walk bounding its infinite face. Given $x, y \in V(D)$, we shall use xDy to denote a subwalk of D from x to y in the clockwise direction. An outerwalk which is a cycle is an *outercycle*.

Obviously, if C is a facial cycle of a 3-connected plane graph G, then (G, C) is a circuit graph. Circuit graphs have the following nice inductive properties. Let (G, C) be a circuit graph such that C is the outercycle. Let C' be a cycle in G and let H be the subgraph of G contained in the closed disc bounded by C'. Then (H, C') is a circuit graph (because if T is a 2-cut of H such that $H - T$ has a component D containing no vertex of C', then by planarity, D is also a component of $G - T$ containing no vertex of C). Also for $v \in V(C)$, $G - v$ is a plane chain of blocks $B_1, \ldots, B_k$ such that v has a neighbor in B_1 and a neighbor in B_k neither of which is a cutvertex of $G - v$. This can be seen as follows. Let x, y be distinct neighbors of v on C such that $xCy = xvy$. Since (G, C) is a circuit graph, every block of $G - v$ contains an edge of yCx. Hence, $G - v$ is a plain chain of blocks along yCx, and each endblock contains x or y.

Let P be a subgraph of a graph G. A *P-bridge* of G is either a single edge of $G - E(P)$ with both ends on P (which is *trivial*), or a component of $G - V(P)$ together with the edges joining the component to P (and all incident vertices). For any P-bridge B of G, the set of *attachments of B on P* is $V(B) \cap V(P)$. We say that P is a *Tutte* subgraph of G if every P-bridge of G has at most three attachments on P. Given a subgraph C of G, we say that P is a *C-Tutte* subgraph of G if P is a Tutte subgraph of G, and every P-bridge of G containing an edge of C has at most two attachments. A *Tutte path* (or *Tutte cycle*) is a path (or cycle) which is a Tutte subgraph.

Note that if (G, C) is a circuit graph and $x, y \in V(C)$ are distinct, then G has at most three $\{x, y\}$-bridges and at most two non-trivial $\{x, y\}$-bridges. Moreover, G has exactly two $\{x, y\}$-bridges if and only if $xy \in E(C)$, or $\{x, y\}$ is a 2-cut of G and $xy \notin E(G)$.

Tutte [5] proved that every 2-connected planar graph contains a Tutte cycle through two given edges on a facial cycle, and deduced that every 4-connected planar graph contains a Hamilton cycle. Thomas and Yu [3] extended these results by finding a Tutte cycle through any three given edges on a facial cycle and deducing that every 4-connected projective planar graph is hamiltonian. We shall show that every circuit graph (G, C) has a Tutte cycle through a given edge of C and two other given vertices. We then use this result to extend the theorem of Whitney [6] that every plane triangulation without separating triangles is hamiltonian. To do this, we define a decomposition of a plane triangulation G into 4-connected 'pieces'. We show that if each piece in the decomposition shares a triangle with at most three other pieces, then G is hamiltonian. A different generalization of Whitney's result was obtained by Dillencourt [2].

2 Preliminary Results

We shall need several lemmas. The first is due to Thomassen [4].

Lemma 1 *Let G be a connected plane graph with outerwalk C. Let $x \in V(C), e \in E(C)$, and $y \in V(G) - \{x\}$ such that G contains a path from x to y through e. Then G contains a C-Tutte path P from x to y through e.*

Lemma 1 is proved in [4] for 2-connected graphs, but it easily generalizes to connected graphs. Note that Lemma 1 implies that every 4-connected planar graph is Hamilton connected.

The following result finds a Tutte path between two vertices through two specified edges.

Lemma 2 *([3], (2.6)) Let G be a 2-connected plane graph with outercycle C, let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in this clockwise order. Then G has a vCu-Tutte path P from u to v using e and f.*

We also need two lemmas concerning the existence of paths avoiding specified vertices.

Lemma 3 *([7], (2.5)) Let G be a connected plane graph with outerwalk C, and let $x, y, z \in V(C)$ be distinct and occur on C in this order such that $G - z$ has a path from x to y. Then $G - z$ has a path P from x to y such that $(P \cup \{z\})$ is an xCz-Tutte subgraph of G.*

Lemma 4 *([3], (2.4)) Let G be a connected plane graph with outerwalk C, and let x, y, u, v be four distinct vertices on C in this clockwise order such that $u, v \notin xCy$. Then $G - \{u, v\}$ contains a path P from x to y such that $(P \cup \{u, v\})$ is an xCy-Tutte subgraph of G.*

3 Tutte Paths Through Specified Vertices

Lemma 5 *Let (G, C) be a circuit graph with C as its outercycle, let x and y be distinct vertices of C, let $e = uv \in E(C)$ such that x, u, v, y occur in this clockwise order around C, and let $z \in V(G)$. Then the following hold.*

(a) G contains a C-Tutte path T from x to y through z.

(b) G contains an xCy-Tutte path T from x to y through e and z, unless either (i) there is a facial cycle D of G other than C and there are distinct vertices $a, b, c, d \in V(C \cap D)$ such that x, a, u, v, b, y, c, d occur on C in this clockwise order, $z \notin \{c, d\}$, and z is contained in the $\{c, d\}$-bridge of G containing cCd; or (ii) there exist vertices $w_1 \in yCx - x$ and $w_2 \in vCy$ such that for both $i \in \{1, 2\}$, no $\{x, w_i\}$-bridge of G contains both z and e; or (iii) there exist vertices $w_1 \in yCx - y$ and $w_2 \in xCu$

such that for both $i \in \{1,2\}$, no $\{y, w_i\}$-bridge of G contains both z and e. In particular, if $xy \in E(C) - \{e\}$ then the exceptional cases (i), (ii) and (iii) cannot occur, if $x = u$ then the exceptional case (i) falls under the exceptional case (ii), and if $y = v$ then the exceptional case (i) falls under the exceptional case (iii).

Proof. We proceed by way of contradiction. Suppose the lemma is false and let G be a counterexample such that $|V(G)|$ is as small as possible.

We first derive a claim, which will be used heavily in the proof of this lemma to modify paths.

Claim 1 *Let (G', C') be a circuit graph with C' as its outercycle. Let $\alpha, \beta\, \nu, \tau$ be vertices occuring in this order on C', $\beta\nu \in E(C')$ and put $L = \alpha C' \beta$. Let H be the plane chain of blocks in $G' - V(L)$ along $\nu C' \tau$. Let D denote the path in the outerwalk of H from τ to ν which contains no edge of $\nu C' \tau$, and let μ be the first vertex on D from τ which is an attachment of some $(H \cup L)$-bridge of G'.*

Let $z \in V(G')$. Define $z_1 \in V(H)$ as follows: if $z \in V(H)$ put $z_1 = z$; if there exist vertices $w \in V(H)$ and $p_1, p_2 \in V(L)$ such that, for $1 \leq i \leq 2$, w and p_i are both attachments of some $(H \cup L)$-bridge of G', and z belongs to the union of $p_1 L p_2$ with those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(p_1 L p_2) \cup \{w\}$, then let $z_1 = w$; otherwise let $z_1 = \nu$.

Suppose that Q is a $\mu D \nu$-Tutte path of H starting at ν and containing z_1, and suppose that $|V(G') - V(Q)| \leq |V(G)| - 2$.

(c) *Suppose $\mu \in V(Q)$. Let $L_\alpha = \alpha L \alpha'$ be the maximal subpath of L such that α' belongs to some $(H \cup L)$-bridge of G' with μ as an attachment. Let F_α be the union of L_α and those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(L_\alpha) \cup \{\mu\}$. Suppose further that, if $z \in F_\alpha$ then, for any vertices $w_1 \in \mu C' \alpha - \mu$ and $w_2 \in \alpha C' \alpha'$ both z and $\alpha' \mu$ are contained in a $\{\mu, w_i\}$-bridge of $F_\alpha + \alpha' \mu$ for some $i \in \{1,2\}$. Then there is a path P in $G' - V(Q)$ from α to β such that $z \in V(P) \cup V(Q)$, $P \cup Q$ is an $\alpha C' \beta$-Tutte subgraph of G', and every Q-bridge of H containing no edge of $\mu D \nu$ is also a $(P \cup Q)$-bridge of G'. Furthermore, if $z \notin F_\alpha - \mu$ then P can be chosen such that $P \cup Q$ is a $\mu C' \beta$-Tutte subgraph of G'.*

(d) *Suppose $\mu \notin V(Q)$. Let X be the Q-bridge of H which contains μ and $L_\alpha = \alpha L \alpha'$ be the maximal subpath of L such that α' belongs to some $(H \cup L)$-bridge of G' with an attachment in $V(X) - V(Q)$. Let F_α be the union of L_α, X, and those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(L_\alpha) \cup (V(X) - V(Q))$. Suppose that $z \notin F_\alpha - V(Q)$. Then there is a path P in $G' - V(Q)$ from α to β such that $z \in V(P) \cup V(Q)$, $P \cup Q$ is an $\alpha C' \beta$-Tutte subgraph of G', and every Q-bridge of H containing no edge of $\mu D \nu$ is also a $(P \cup Q)$-bridge of G'.*

Proof of (c). Since G' is a plane graph, we can divide L into non-trivial edge-disjoint subpaths pLq of three types according to the Q-bridges of H: (type 1) for some Q-bridge B of H, pLq is a maximal subpath of L such that p and q belong to some $(H \cup L)$-bridges of G' each with an attachment contained in $V(B) - V(Q)$; (type 2) for some vertex v of Q, pLq is the maximal subpath of L such that both p and q belong to some $(H \cup L)$-bridges of G' with v as an attachment; (type 3) the remaining non-trivial edge maximal subpaths of L.

Obviously, each edge of L belongs to exactly one of these subpaths. We label these subpaths (from α to β) as L_i, $i = 1, \cdots, m$. Let the endvertices of L_i be x_i and x_{i+1} such that $x_1, x_2, \ldots, x_{m+1}$ occur on C' in this order, where $x_1 = \alpha$ and $x_{m+1} = \beta$.

Suppose L_i is of type 1. Let F_i be the union of L_i, B, and those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(L_i) \cup (V(B) - V(Q))$. By Lemma 4, $F_i - (V(B) \cap V(Q))$ has a path P_i from x_i to x_{i+1} such that $P_i \cup (Q \cap B)$ is an $x_i C' x_{i+1}$-Tutte subgraph of F_i. Because $z_1 \in V(Q)$, $z \notin F_i - (V(Q) \cap V(B))$.

Suppose L_i is of type 2. Let F_i be the union of L_i and those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(L_i) \cup \{v\}$. Let C_i be the outerwalk of F_i. Applying (b) inductively to $(F_i + x_{i+1}v, vC_ix_{i+1}v)$ for $1 \leq i < m$, we find an $x_iC_ix_{i+1}v$-Tutte path P_i from x_i to v through $x_{i+1}v$, and also through z if $z \in V(F_i)$. Similarly, if $i = m$ we apply (b) inductively to $(F_m + \nu x_m, \nu x_m C_m \nu)$ we find a $\nu x_m C_m \beta$-Tutte path P_m from ν to β through νx_m, and also through z if $z \in V(F_m)$. Note that the exceptional cases of (b) cannot occur by the internal 3-connectivity of G' for $1 < i < m$, by the hypothesis on w_1, w_2 in (c) and the fact that $x_{i+1}\mu \in E(F_i + x_{i+1}\mu)$ for $i = 1$, and by the fact that $\nu\beta \in E(C_m) - \{\nu x_m\}$ for $i = m$.

Suppose L_i is of type 3. Let F_i be the union of L_i and those $(H \cup L)$-bridges of G' whose attachments are all contained in $V(L_i)$. If $F_i = x_i x_{i+1}$ let $P_i = x_i x_{i+1}$. Otherwise chose $z_i \in V(F_i) - \{x_i, x_{i+1}\}$ with $z_i = z$ if $z \in V(F_i)$. Applying (a) inductively to $(F_i + x_i x_{i+1}, x_i L_i x_{i+1} x_i)$ we find an $x_i L_i x_{i+1} x_i$-Tutte path P_i from x_i to x_{i+1} through z_i.

It is now easy to see that $P := \bigcup_{i=1}^{m} P_i$ is the desired path in $G' - V(Q)$.

To prove the second assertion of (c) we proceed similarly, but instead of using (b) inductively to deal with $L_1 = L_\alpha$, we use Lemma 3 to find a path P_1 in $F_1 - \mu$ from α to x_2 such that $P_1 \cup \{\mu\}$ is a $\mu C' x_2$-Tutte subgraph of F_1.

Proof of (d). We proceed as in (c), with the exception that we use Lemma 4 to construct a path P_1 in $F_1 - (V(Q) \cap V(X)) := F_\alpha - (V(Q) \cap V(X))$ such that $P_1 \cup (Q \cap X)$ is an $\alpha C \alpha'$-Tutte subgraph of F_1. ∎

Claim 2 *(a) holds for G.*

Suppose the claim is false. If $z \in V(C)$ then (a) holds for G by Lemma 1. Hence $z \notin V(C)$.

Claim 2.1 G is 3-connected.

Proof. Suppose $\{s, t\}$ is a 2-cut of G. We may suppose that $\{s, t\}$ has been chosen such that the $\{s, t\}$-bridge B which contains z is minimal. Let the outerwalk of B be denoted by $tCs'sFt$. Let $J := G - (V(B) - \{s, t\})$.

We first consider the case when $x, y \in V(B)$. Applying (a) inductively to the circuit graph $(B + st, tCst)$ we deduce that $B + st$ has a tCs-Tutte path P from x to y through z. If $st \notin E(P)$ then $T := P$ satisfies (a) for G. Hence $st \in E(P)$. By Lemma 1 we find an sCt-Tutte path P' from s to t in J. Then $T := (P - st) \cup P'$ satisfies (a) for G.

We next consider the case when $x, y \in V(J)$. Applying (a) inductively to the circuit graph $(B + st, tCst)$ we deduce that $B + st$ has a tCs-Tutte path P from s to t through z. By Lemma 1 we find an sCt-Tutte path P' in $J + st$ from x to y through st. Then $T := (P' - st) \cup P$ satisfies (a) for G.

Finally we consider the case when $\{s, t\}$ separates x and y. By symmetry we may assume that $y \in V(B)$ and $x \in V(J)$. By Lemma 1 we find an sCt-Tutte path P' in $J + st$ from x to s through ts. Let $L := yCs'$, let H be the plane chain of blocks in $B - V(L)$ along sFt, and let W denote the path in the outerwalk of H from t to s which contains no edge of F. Let μ be the first vertex on W from t which is an attachment of some $(H \cup L)$-bridge of B. Define $z_1 \in V(H)$ as follows: if $z \in V(H)$ put $z_1 = z$; if there exist vertices $w \in V(H)$ and $p_1, p_2 \in V(L)$ such that, for $1 \leq i \leq 2$, w and p_i are both attachments of some $(H \cup L)$-bridge of G and z belongs to the union of $p_1 C p_2$ with those $(H \cup L)$-bridges of G whose attachments are all contained in $V(p_1 C p_2) \cup \{w\}$, then let $z_1 = w$; otherwise let $z_1 = s$. Let f be an edge of W incident with μ. Applying (b) inductively to the circuit graph $(H + st, tWst)$ and because $st \in E(tWst) - \{f\}$, we deduce that $H + st$ has a tWs-Tutte path Q from s to t through z_1 and f (and hence μ). Let $L_1 = yLy'$ be the maximal subpath of L such that y' belongs to some $(H \cup L)$-bridge of B with μ as an attachment. Then z does not belong to the union of those $(H \cup L)$-bridges of B whose attachments are all contained in $V(L_1) \cup \{\mu\}$, otherwise (because $z \notin V(C)$) $\{\mu, y'\}$ would contradict the choice of $\{s, t\}$ to minimize the $\{s, t\}$-bridge B containing z. Applying Claim 1(c) (with $B, tCsFt, y, y', s', s, t$ as $G', C', \alpha, \alpha', \beta, \nu, \tau$, respectively), we obtain a path P from s' to y in $B - V(Q)$ such that $P \cup Q$ is a $\mu Cs'$-Tutte subgraph of B and every Q-bridge of H containing no edge of tWs is also a $(P \cup Q)$-bridge of B. Then $T := xP'tQss'Py$ satisfies (a) for G. ∎

We can now complete the proof of Claim 2. We may assume by symmetry that xCy has at least three vertices. Let $C = xCy''y'yCx'x$ and $J := G - V(xCy'')$, where $y' \neq y$ but we may have $x = y''$ or $x' = y$ or both. Let H be the plane chain of blocks in J along $y'Cx'$. Since G is 3-connected, H is a block. Let the outerwalk of H be $y'Cx'Fy'$. Hence either $(H, y'Cx'Fy')$ is a circuit graph, or else $x' = y$ and $H = y'x'$. Define $z_1 \in V(H)$ as follows: if $z \in V(H)$ put $z_1 = z$; if there exist vertices $w \in V(H)$ and $p_1, p_2 \in V(xCy'')$ such that, for $1 \leq i \leq 2$, w and p_i are both attachments of some $(H \cup xCy'')$-bridge of G, and z belongs to the union of $p_1 C p_2$ with those $(H \cup xCy'')$-bridges of G whose attachments are all contained in $V(p_1 C p_2) \cup \{w\}$, then let $z_1 = w$; otherwise let $z_1 = y'$. If $H = y'x'$ then let $Q = H$. Otherwise we may apply (b) inductively to $(H, y'Cx'Fy')$ to deduce that H has a $y'Cx'Fy'$-Tutte path Q from y to y' through z_1 and an edge f of yCx' incident to x'. Note that the exceptional cases of (b) do not occur by the fact that $yy' \in E(y'Cx'Fy') - \{f\}$. Applying Claim 1(c) (with G, C, x, y'', y', x' as $G', C', \alpha, \beta, \nu, \tau = \mu$, respectively) and using Claim 2.1 to show that the hypothesis in (c) on w_1, w_2 must hold, we obtain a path P in $G - V(Q)$ from y'' to x such that $z \in V(P) \cup V(Q)$, $P \cup Q$ is an xCy''-Tutte subgraph of G, and every Q-bridge of H containing no edge of $x'Fy'$ is also a $(P \cup Q)$-bridge of G. Then $T := yQy'y''Px$ satisfies (a) for G. ∎

Claim 3 *(b) holds for G.*

Proof. Suppose $z \in V(xCy)$. Then by Lemma 2, G has an xCy-Tutte path T from x to y through e and z, and so, (b) holds for G. So assume $z \notin V(xCy)$.

We claim that either (1) $y \neq v$ and z belongs to a block of $G - V(xCu)$ which contains an edge of vCy or (2) $x \neq u$ and z belongs to a block of $G - V(vCy)$ which contains an edge of xCu. Suppose neither (1) nor (2) holds. If $u = x$ and $v = y$ then G belongs to the exceptional cases (ii) and (iii) of (b). Assume $x = u$ and $y \neq v$. Then there is a vertex $w_1 \in yCx - x$ such that $\{x, w_1\}$ is a 2-cut of G separating z from e, or else (1) holds. Let $w_2 = v$. Then for each $i \in \{1, 2\}$ no $\{x, w_i\}$-bridge of G contains both z and e, and we have exceptional case (ii) of (b). Similarly, if $x \neq u$ and $y = v$, then exceptional case (iii) of (b) occurs. So assume that $x \neq u$ and $y \neq v$. Let K denote the plane chain of blocks in $G - xCu$ along vCy. Since (1) does not hold, there exists some $(K \cup xCu)$-bridge B of G such that $z \in B - V(xCu \cup K)$. Let $c \in V(B) \cap V(K)$. Then $c \in yCx - x$; otherwise, (2) holds because (G, C) is a circuit graph. Let $a \in V(B) \cap V(xCu)$ such that aCu is minimal. Then $\{a, c\}$ is a 2-cut of G separating e from z. By a symmetric argument, we may assume that G has a 2-cut $\{b, d\}$ separating e from z, where $b \in vCy$ and $d \in yCx - y$. Clearly, $a \neq b$, $z \notin \{c, d\}$, and $c \in yCd$. If $a = d$ then exceptional case (ii) occurs, and if $c = b$ the exceptional case (iii) occurs. So $a \neq d$ and $c \neq b$. By planarity, we have exceptional case (i) of (b).

Hence we may assume by symmetry that (1) holds. Let $L = xCu$ and H be the plane chain of blocks in $G - V(L)$ along vCy. By (1), $z \in V(H)$. Define $z_1 = z$. Let F be the path in the outerwalk of H from y to v which contains no edge of vCy. Applying (a) inductively to each block of H we obtain a $vCyFv$-Tutte path Q in H from v to y through z_1. Let y' be the first vertex of F from y which is an attachment of some $(H \cup L)$-bridge of G. If $y' \in V(Q)$, let $L_x = xLx'$ be the maximal subpath of L such that x' belongs to some $(H \cup L)$-bridge of G with y' as an attachment, and F_x be the union of L_x with those $(H \cup L)$-bridges of G whose attachments are all contained in $V(L_x) \cup \{y'\}$. If $y' \notin V(Q)$ let X be the Q-bridge of H which contains y', $L_x = xLx'$ be the maximal subpath of L such that x' belongs to some

$(H \cup L)$-bridge of G with an attachment in $V(X) - V(Q)$, and F_x be the union of L_x, X and those $(H \cup L)$-bridges of G whose attachments are all contained in $V(L_x) \cup (V(X) - V(Q))$. Since $z \in V(Q)$, $z \notin F_x - y'$. Then by Claim 1 (c) and (d) (with G, C, x, x', u, v, y, y' as $G', C', \alpha, \alpha', \beta, \nu, \tau, \mu$, respectively), we can find a path P from u to x in $G - V(Q)$ such that $z \in V(P \cup Q)$, $P \cup Q$ is a $y'Cu$-Tutte subgraph of G, and every Q-bridge of H containing no edge of $y'Fv$ is also a $(P \cup Q)$-bridge of G. Then $T := xPuvQy$ satisfies (b) for G. ∎

Lemma 6 *Let (G, C) be a circuit graph, r, z be vertices of G and $e \in E(C)$. Then G contains a C-Tutte cycle X through e, r, and z.*

Proof. Suppose the lemma is false and let G be a counterexample chosen such that $|V(G)|$ is as small as possible. If either r or z lies on C then the lemma holds by Lemma 5(b) (because $xy \in E(C) - \{f\}$, where $f \in E(C) - \{xy\}$ and f is incident to $\{r, z\} \cap V(C)$). Hence $\{r, z\} \cap V(C) = \emptyset$. Without loss of generality, assume that C is the outercycle of G.

Claim 4 *G is 3-connected.*

Proof. Let $\{s, t\}$ be a 2-cut of G and B_1 and B_2 be the non-trivial $\{s, t\}$-bridges of G such that $sCt \subset B_1$. We first suppose that r is in B_1 and e, z are in B_2. By Lemma 5(a) (with $B_1 + st, sCts, s, t$ as G, C, x, y, respectively), $B_1 + st$ has an $sCts$-Tutte path P_1 from s to t through r. By Lemma 5(b) (with $B_2 + st, tCst, s, t$ as G, C, y, x, respectively) and because $st \in E(tCst) - \{e\}$, $B_2 + st$ has a $tCst$-Tutte path P_2 from s to t through e and z. Then $X := P_1 \cup P_2$ is the required Tutte cycle in G. Thus we may assume that $\{s, t\}$ does not separate r and z.

We next suppose that e is in B_1 and r, z are in B_2. By Lemma 1, $B_1 + st$ has an $sCts$-Tutte path P_3 from s to t through e. By induction, $B_2 + st$ has a $tCst$-Tutte cycle T_1 through st, r and z. Then $X := P_3 \cup (T_1 - st)$ is the required Tutte cycle in G.

Finally we suppose that e, r, z are in B_2. By induction, $B_2 + st$ has a $tCst$-Tutte cycle T_2 through e, r, z. If $st \notin E(T_2)$ then T_2 is the required Tutte cycle in G. Hence $st \in E(T_2)$. By Lemma 1, B_1 has an $sCts$-Tutte path P_4 from s to t. Then $X := (T_2 - st) \cup P_4$ is the required Tutte cycle in G. ∎

Claim 5 *$G - V(C)$ is connected.*

Proof. This follows immediately from Claim 4 and planarity. ∎

Let H be a block of $G - V(C)$ and W be the outerwalk of H. By Claim 5, every $(H \cup C)$-bridge of G has exactly one attachment on H. Let $w_1, w_2, \ldots, w_m$ be the attachments on H of the $(H \cup C)$-bridges of G in clockwise order around W. Let $B_{i,1}, B_{i,2}, \ldots, B_{i,n_i}$ be the $(H \cup C)$-bridges of G which contain w_i for $1 \leq i \leq m$. Let $s_{i,j}Ct_{i,j}$ be the minimal segment of C such that $B_{i,j} \cap C \subseteq s_{i,j}Ct_{i,j}$ and $B \cap s_{i,j}Ct_{i,j} \subseteq \{s_{i,j}, t_{i,j}\}$ for all $(H \cup C)$-bridges $B \neq B_{i,j}$ of G. Let $F_{i,j} := B_{i,j} \cup s_{i,j}Ct_{i,j}$.

Claim 6 *We may choose H such that $|F_{i,j} \cap \{e, r, z\}| \leq 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$.*

Proof. Choose H such that either $|F_{i,j} \cap \{e, r, z\}| \leq 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$, or, if this is not possible, such that if $|F_{i,j} \cap \{e, r, z\}| \geq 2$ then $F_{i,j}$ is minimal. Suppose $|F_{i,j} \cap \{e, r, z\}| \geq 2$. Let H' be the block of $G - V(C)$ with $w_i \in H' \subseteq F_{i,j}$. Then, in an obvious notation, we have that if $|F'_{k,\ell} \cap \{e, r, z\}| \geq 2$ then $F'_{k,\ell}$ is a proper subgraph of $F_{i,j}$. This would contradict the choice of H and hence Claim 6 holds. ∎

We now continue with the proof of the lemma. Let $t_i = t_{i,n_i}$ for $1 \leq i \leq m$. We may suppose that the labeling has been chosen such that $e \in t_mCt_1$. Choose (k, ℓ) to be as small as possible in the lexicographic ordering such that $2 \leq k \leq m$, $1 \leq \ell \leq n_k$, and $t_{k,\ell} \neq t_1$. Note that (k, ℓ) exists since otherwise $\{t_1, w_1\}$ would be a 2-cut in G, contradicting Claim 4. Note also that Claim 4 implies that if $(k, \ell) > (2, 1)$ then $V(F_{i,j}) = \{t_1, w_i\}$ for $(2, 1) \leq (i, j) < (k, \ell)$. For $u \in \{r, z\}$, define $u_1 \in H$ as follows: if $u \in H$ put $u_1 = u$; if $u \in F_{i,j}$ for some $1 \leq i \leq m$ and $1 \leq j \leq n_j$, put $u_1 = w_i$. Let H' be the graph obtained from H by adding the vertex t_1 and edges t_1w_i, $i = 1, \ldots, k$ and let $C' := w_kWw_1t_1w_k$ be the outercycle of H'. By induction there exists a C'-Tutte cycle Q' in H' through t_1w_k, r_1, z_1. Note that $w_1 \in Q'$ since otherwise the Q'-bridge of H' containing w_1 would contain an edge of C', and would have at least three attachments on Q' (the vertex t_1 and at least two attachments on H since H is a block). Let $Q := Q' - t_1$. Then $w_1 \in Q$ and Q is a path from w_j to w_k in H for some $j \in \{1, \ldots, k-1\}$.

Since G is a plane graph, we can divide C into non-trivial edge-disjoint subpaths sCt of three types according to the Q-bridges of H: (i) for some

Q-bridge B of H, sCt is a maximal subpath of C such that s and t belong to some $(H \cup C)$-bridges of G each with an attachment in $V(B) - V(Q)$; (ii) for some vertex v of Q and some $(H \cup C)$-bridge B of G with v as an attachment, sCt is the maximal subpath of C such that both s and t belong to B (in this case $sCt = s_{p,q}Ct_{p,q}$ for some $s_{p,q} \neq t_{p,q}$); (iii) the remaining edges of C.

Each edge of C belongs to exactly one of these subpaths. We label these subpaths as C_i, $i = 1, \cdots, b$. Let the endvertices of C_i be x_i and x_{i+1} such that $x_1, \ldots, x_b$ are on C in this clockwise order, and subscripts are read modulo b. Since $w_1, w_k \in Q$ we may choose the labeling such that $x_1 = s_{1,n_1}$ and $x_{h+1} = t_{k,\ell}$ for some $h \geq 1$.

We first consider intervals C_i of type (i). Let J_i be the union of C_i, B, and those $(H \cup C)$-bridges of G whose attachments are all contained in $V(C_i) \cup (V(B) - V(Q))$. By Lemma 4, $J_i - (V(B) \cap V(Q))$ has a path P_i from x_i to x_{i+1} such that $P_i \cup (B \cap Q)$ is a C_i-Tutte subgraph of J_i. Note that e, r, z do not belong to $J_i - (V(B) \cap V(Q))$ by the facts that $e \in t_m C t_1$ and $w_1, r_1, z_1 \in V(Q)$.

We next consider intervals C_i of type (ii). Then $C_i = s_{p,q}Ct_{p,q}$ for some $1 \leq p \leq m$ and $1 \leq q \leq n_p$. Let $J_i := F_{p,q}$, and let W_i be the outerwalk of J_i. First assume that $h+1 \leq i \leq b$ or $i = 1$ and $w_j \neq w_1$. Since $|J_i \cap \{e, r, z\}| \leq 1$ by Claim 6 and because $x_i w_p \in E(w_p x_i W_i x_{i+1} w_p) - \{x_{i+1} w_p\}$, we may apply Lemma 2 (when $e \in E(J_i)$) or Lemma 5(b) (when $\{r, z\} \cap V(J_i) \neq \emptyset$) to $(J_i + \{x_{i+1}w_p, x_i w_p\}, w_p x_i W_i x_{i+1} w_p)$ to find an $x_i W_i x_{i+1} w_p$-Tutte path P_i' from x_i to w_p through $x_{i+1}w_p$ and $J_i \cap \{e, r, z\}$. Let $P_i = P_i' - w_p$ if $i \neq 1$, and otherwise, let $P_1 = x_1 P_1' x_2 w_j$ (when $i = 1$ and $w_j \neq w_1$, we have $x_1 = s_{1,n_1}$ and $x_2 = t_{1,n_1} = t_1$, and so $x_2 w_j \in E(G)$). Now assume that $i = 1$ and $w_j = w_1$ (in this case, $t_1 = x_2$, and so, $h \geq 2$ because $x_{h+1} = t_{k,\ell} \neq t_1$). Because $w_1 x_1 \in E(x_1 W_1 w_1 x_1) - \{f\}$, where $f \in E(W_1)$ and f is incident to x_2, we apply Lemma 2 (when $e \in E(J_1)$) or Lemma 5(b) (when $\{r, z\} \cap V(J_1) \neq \emptyset$) to $(J_1 + w_1 x_1, x_1 W_1 w_1 x_1)$ to find an $x_1 W_1 w_1$-Tutte path P_1 from x_1 to w_1 through f and $J_1 \cap \{e, r, z\}$. Finally assume that $i = h$. Because C_i is of type (ii), $s_{k,\ell} \neq t_{k,\ell}$. Hence $s_{k,\ell} = x_h$. Note that $e \notin J_h$ because $e \in t_m C t_1$ and $C_h \subset t_1 C t_k$. Suppose $x_h = t_1$. Let $J_h' := (J_h - x_h) + x_{h+1} w_k$. Assume that the edge $x_{h+1} w_k$ is added so that $w_k W_h x_{h+1} w_k$ is the outercycle of J_h. Let F_h denote the outercycle of J_h', and let $f \in E(F_h) - \{x_{h+1} w_k\}$ such that f is incident to the neighbor of x_h on $C \cap J_h'$. Because G is 3-connected, (J_h', F_h) is a circuit graph. Applying Lemma 5(b) to (J_h', F_h) and because

$x_{h+1}w_k \in E(F_h) - \{f\}$, we find a $w_kF_hx_{h+1}$-Tutte path P_h from w_k to x_{h+1} through f and $J_h \cap \{r, z\}$. Now assume $x_h \neq t_1$. Let $f \in E(W_h)$ such that f is incident to x_h. Because $w_kx_{h+1} \in E(w_kW_hx_{h+1}w_k) - \{f\}$, we may apply Lemma 5(b) to $(J_h + x_{h+1}w_k, w_kW_hx_{h+1}w_k)$ to find a $w_kW_hx_{h+1}$-Tutte path P_h from w_k to x_{h+1} through f and $J_h \cap \{r, z\}$.

Finally, for intervals C_i of type (iii), let $P_i := C_i$.

Note that if $s_{k,\ell} = t_{k,\ell}$ then $V(F_{k,\ell}) = \{w_k, x_{h+1}\}$, and if $s_{1,n_1} = t_{1,n_1}$ then $V(F_{1,n_1}) = \{w_1, x_1\}$. We define X as follows. If $s_{1,n_1} = t_{1,n_1}$ and $s_{k,l} = x_{h+1}$, then let $X := w_jQw_kx_{h+1}P_{h+1}\dots P_bx_1w_j$. If $s_{1,n_1} = t_{1,n_1}$ and $s_{k,l} \neq x_{h+1}$, then let $X := w_jQw_kP_hx_{h+1}P_{h+1}\dots P_bx_1w_j$. If $s_{1,n_1} \neq t_{1,n_1}$ and $s_{k,l} = x_{h+1}$, then let $X := w_jQw_kx_{h+1}P_{h+1}\dots P_bx_1P_1w_j$. If $s_{1,n_1} \neq t_{1,n_1}$ and $s_{k,l} \neq x_{h+1}$, then let $X := w_jQw_kP_hx_{h+1}P_{h+1}\dots P_bx_1P_1w_j$.

It is then straightforward to verify that X is the required Tutte cycle in G. ∎

4 Plane Triangulations

We first use Lemma 6 to show that we can find a Hamilton cycle through edges in specified triangles in a plane triangulation which has no separating triangles.

Theorem 7 *Let G be a plane triangulation with no separating triangles. Let T, T_1, T_2 be distinct triangles in G. Let $V(T) = \{u, v, w\}$. Then there exists a Hamilton cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$ such that uv, uw, e_1, e_2 are distinct and contained in $E(C)$.*

Proof. Let $H = G - u$ and F be the face of H which contained u. We may suppose that F is the infinite face of H. Let $X_i = T_i \cap H$ for $1 \leq i \leq 2$. Then either $X_i = T_i$ or X_i is an edge incident with F. Let H' be the graph obtained from H as follows: if X_i is an edge of H then subdivide X_i with a new vertex z_i; if $X_i = T_i$ then in the face of H bounded by T_i insert a new vertex z_i and join z_i to each vertex of T_i. Let F' be the outercycle of H'. Then (H', F') is a circuit graph. By Lemma 6, H' has an F'-Tutte cycle C' through vw, z_1 and z_2. Using the facts that the only possible separating triangles in H' are T_1 and T_2 and that C' is an F'-Tutte cycle in H', we deduce that C' is a Hamilton cycle in H'. Let C be the cycle obtained from

$C' - \{vw, z_1, z_2\}$ by adding uv, uw and edges e_i between the neighbours of z_i on C' for $1 \leq i \leq 2$. Then C is the required Hamilton cycle in G. ∎

We next apply Theorem 7 to general plane triangulations. To accomplish this we first need to describe how to decompose a plane triangulation into 4-connected 'pieces'. Let G be a plane triangulation. Suppose that T is a separating triangle in G. Then we may separate G into two graphs G_1 and G_2 such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = T$ and both G_1 and G_2 have at least four vertices. Then G_i is a plane triangulation and T is a facial triangle of G_i for $1 \leq i \leq 2$. We shall refer to T as a *marker triangle* in G_1 and G_2. We now iterate this procedure for both G_1 and G_2. We continue until we obtain a collection S of plane triangulations each of which has no separating triangles. We shall refer to the graphs in S as *pieces* of G. Note that each separating triangle of G will occur as a marker triangle in exactly two pieces of G. We define a new graph D whose vertices are the pieces in S, and in which two pieces are joined by an edge if they have a marker triangle in common. It follows from the decomposition theory developed by Cunningham and Edmonds [1] that D is a tree and also that the set S and the tree D are uniquely defined by G. We shall refer to D as *the decomposition tree* of G.

Theorem 8 *Let G be a 3-connected plane triangulation. Suppose that G has a decomposition tree D of maximum degree at most three. Let H be a piece of G corresponding to a vertex of D of degree at most two, T be a facial cycle of both H and G, and $V(T) = \{u, v, w\}$. Then G has a Hamilton cycle through uv and uw.*

Proof. We proceed by induction on $|V(D)|$. If D has only one vertex then the result follows from Theorem 7. Hence we may assume that D has at least two vertices. To simplify notation we shall suppose that H has degree two in D. (A similar but simpler proof holds if H has degree one.) Let T_1 and T_2 be the marker triangles contained in H. By Theorem 7, H has a Hamilton cycle C which contains distinct edges uv, uw, $f_1 \in E(T_1)$ and $f_2 \in E(T_2)$.

For $1 \leq i \leq 2$, let $V(T_i) = \{u_i, v_i, w_i\}$ where $f_i = v_i w_i$. Let the pieces of G separated by T_i be G_i, G_i' where $G_i \cap H = T_i = G_i \cap G_i'$. Let D_1 and D_2 are the two components obtained by deleting the vertex H from the decomposition tree D, labeled in such a way that T_i is contained in a piece H_i of G which is a vertex of D_i. Then D_i is the decomposition tree for

G_i, and H_i has degree at most two in D_i. Thus we may apply induction to G_i and find a Hamilton cycle C_i in G_i through u_iv_i and u_iw_i. Then $(C - \{v_1w_1, v_2w_2\}) \cup (C_1 - u_1) \cup (C_2 - u_2)$ is the required Hamilton cycle in G. ∎

The hypothesis of Theorem 8 that D has maximum degree at most three cannot be weakened to maximum degree at most four. To see this we define a sequence of graphs recursively by putting $G_1 = K_4$ and, for $i \geq 1$, letting G_{i+1} be the graph obtained from G_i by inserting a new vertex into each face of G_i then joining the new vertex to every vertex incident with the face. Then G_i has a decomposition tree of maximum degree four for $i \geq 2$, and is non-hamiltonian for $i \geq 3$.

One may hope to extend Theorem 8 to an arbitrary 3-connected planar graph G. We can define a decomposition of G into 4-connected pieces in a similar way to the above decomposition for triangulations, the only difference being that, when we separate G along a 3-vertex cut, we add three new edges between the vertices of the cut on each side of the separation, to form the 'marker triangle'. One drawback of this approach is that the decomposition need no longer be unique. The main drawback, however, is that the obvious extension of Theorem 8 is false. To see this consider the Herschel graph H of Figure 1. If we separate H along the three 3-cuts $\{2, 4, 9\}, \{1, 6, 11\}, \{3, 8, 10\}$ we obtain a decomposition of H into four pieces, G_1, G_2, G_3, G_4, where G_1 and G_4 are isomorphic to K_4, and G_2 and G_3 are isomorphic to the octahedron. The corresponding decomposition tree is the path $G_1G_2G_3G_4$ of length three, which has maximum degree two. The graph H is not hamiltonian, however, since it is bipartite and has an odd number of vertices.

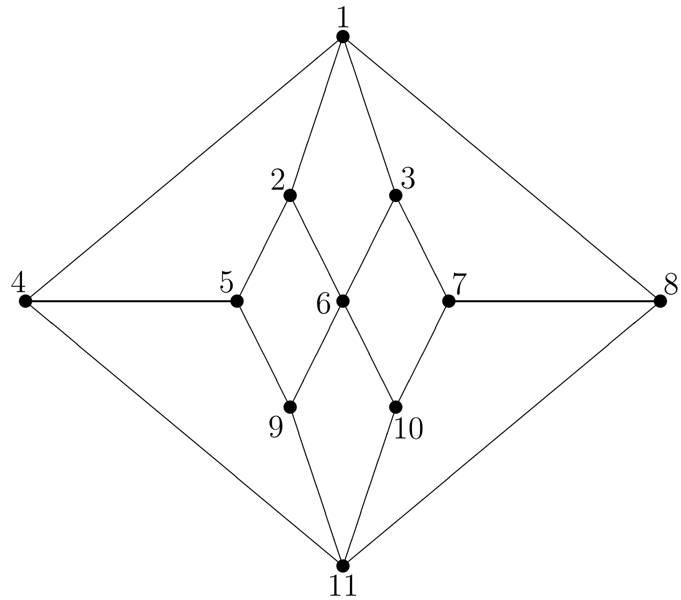

Figure 1

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