The Circumference of a Graph with no $K_{3,t}$-minor, II

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Abstract

The class of graphs with no $K_{3,t}$-minors, $t \geq 3$, contains all planar graphs and plays an important role in graph minor theory. In 1992, Seymour and Thomas conjectured the existence of a function $\alpha(t) > 0$ and a constant $\beta > 0$, such that every 3-connected n-vertex graph with no $K_{3,t}$-minors, $t \geq 3$, contains a cycle of length at least $\alpha(t)n^{\beta}$. The purpose of this paper is to confirm this conjecture with $\alpha(t) = (1/2)^{t(t-1)}$ and $\beta = \log_{1729} 2$.

*Supported in part by NSA and by NSFC Project 10628102

†Supported in part by the Research Grants Council of Hong Kong and Seed Funding for Basic Research of HKU.

1 Introduction

Let G be a graph. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle in G. The problem of determining $c(G)$ is a classical NP-hard problem, so the focus of extensive research has been on the lower bound of $c(G)$. While studying paths in polytopes, Moon and Moser [11] implicitly conjectured that for every 3-connected planar graph G on n vertices, $c(G) = \Omega(n^{\log_3 2})$; Grünbaum and Walther [8] later made (explicitly) the same conjecture for 3-connected cubic planar graphs. Over the past four decades various authors have obtained several theorems related to the Moon-Moser conjecture; see, for instance, [7, 9]. This conjecture was eventually established by Chen and Yu [5], where the same bound (within a constant factor) was also derived for 3-connected graphs embeddable in the torus or the Klein bottle. In [3] this result was applied to prove that $c(G) \geq \epsilon(g)n^{\log_3 2}$ for every 3-connected n-vertex graph G of orientable genus g, where $\epsilon(g)$ is a positive function dependent on g; we refer to [14] for an improved bound (with a positive constant in place of $\epsilon(g)$) for "locally planar" graphs.

It is well known that a 3-connected graph with no $K_{3,3}$-minor is planar, with the exception of K_5. So the result obtained by Chen and Yu [5] can be extended to graphs with no $K_{3,3}$-minors; and thus a natural question is to ask whether a similar result holds for graphs with no $K_{3,t}$-minors, where $t \geq 4$. As discovered by Robertson and Seymour [13], the class of graphs with no $K_{3,t}$-minors plays an important role in graph minor theory: if a minor-closed class of graphs does not contain all graphs, then every graph in it is glued together in a tree-like fashion from graphs that can almost be embedded in a fixed surface. Moreover, if a graph is embeddable in a given surface, then it contains no $K_{3,t}$ as a minor for some $t > 0$; see Lovász [10] for a comprehensive survey of graph minor theory. Although a structural characterization of all graphs with no $K_{3,t}$-minors is still unavailable and seems extremely hard to obtain, Oporowski, Oxley, and Thomas [12] proved that if a 3-connected graph contains no $K_{3,t}$-minors, then it must contain a large wheel. Motivated by this result, Thomas and Seymour [15] made the following two conjectures.

Conjecture 1.1. (Thomas) *There exist two functions $\alpha(t) > 0$ and $\beta(t) > 0$ such that, for any integer $t \geq 3$ and any 3-connected n-vertex graph G with no $K_{3,t}$-minor, $c(G) \geq \alpha(t)n^{\beta(t)}$.*

Conjecture 1.2. (Seymour and Thomas) *There exist a function $\alpha(t) > 0$ and a constant $\beta > 0$ such that, for any integer $t \geq 3$ and any 3-connected n-vertex graph G with no $K_{3,t}$-minor, $c(G) \geq \alpha(t)n^{\beta}$.*

Jointly with Sheppardson, we have obtained a proof of Conjecture 1.1; see [4]. The purpose of this paper is to confirm the second one.

Theorem 1.3. *Let G be a 3-connected n-vertex graph with no $K_{3,t}$-minor. Then $c(G) \geq (1/2)^{t(t-1)}n^{\log_{1729} 2}$.*

The base 1729 is chosen because it is the best we can do in the proof of Claim 7.11; it would be interesting to know if this bound is best possible. Nevertheless, we strongly believe that the above result can be strengthened further if G enjoys higher connectivity.

Conjecture 1.4. *There exists a function $\alpha(t) > 0$ such that, for any integer $t \geq 4$ and any 4-connected n-vertex graph G with no $K_{3,t}$-minor, $c(G) \geq \alpha(t)n$.*

Böhme, Maharry, and Mohar [2] have proved that every 7-connected graph with a sufficiently large number of vertices contains a $K_{3,t}$-minor for any fixed positive integer t; so to attack Conjecture 1.4 one may start with 6-connected graphs. The proof of Theorem 1.3 relies heavily on Tutte's algorithm [20] for decomposing 2-connected graphs into 3-blocks. We envisage that the most difficult step in a proof of Conjecture 1.4, if any, might be to find a counterpart of Tutte's algorithm for decomposing 3-connected graphs into 4-blocks. In Tutte's decomposition, the 3-blocks involved form a tree-like structure, yet the situation for higher connectivity seems dramatically different.

The study of the longest cycle problem on 4-connected planar graphs dates back to 1931 when Whitney [21] proved that every 4-connected plane triangulation contains a Hamiltonian cycle; this work was obviously motivated by Tait's theorem on face 4-colorablility of Hamiltonian plane graphs. Whitney's theorem [21] has been generalized to all 4-connected planar graphs by Tutte [19] and further to all 4-connected projective-planar graphs and 5-connected toroidal graphs by Thomas and Yu [16, 17]; related work can also be found in [18]. Generalizing to other surfaces, Yu [22] showed that every "locally planar" 5-connected triangulation of a surface contains a Hamiltonian cycle. Conjecture 1.4, made in a more general setting, is in the same spirit as that of previous work.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic terminology and notations, present a variant of Tutte's algorithm for decomposing 2-connected graphs into 3-blocks, define two index functions θ and ϕ, and formulate the main theorem consisting of three separate statements in terms of θ and ϕ. In Section 3, we deal with rooted $K_{3,t}$-minor, and show that if a 3-connected graph has a $K_{3,t}$-minor, then it contains a $K_{3,\lceil t/3\rceil}$-minor rooted at any three given vertices. Based on this result, we can not only merge minors in different parts of the graph to form a larger minor as desired but have a good control of these minors as well. We also recall some useful properties of the function $f(x) = x^{\log_b 2}$ from [4], which allow us to discard some parts of the graph in our search procedure. In Section 4, we study the longest cycle problem on graphs with weights on edges; in our proof we shall use weights to keep track of the lengths of paths generated in 3-blocks or some of their unions. Finally, in Sections 5-7, we establish the three technical statements stated in Section 3, respectively.

2 Preliminaries

We start this section with some basic terminology and notations.

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. Set $|G| := |V(G)|$; we call it the *size* of G. For each $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by U. We call U a *connected set* of G if $G[U]$ is connected. We shall use G/U to denote the graph obtained from G by contracting U (and deleting the resulting multiple edges and loops) if U is a connected set. Throughout this paper, we set $G - U := G[V(G) - U]$ and set $G - u := G - U$ if $U = \{u\}$. We say that U is a *cutset* of G if G is connected and $G - U$ is disconnected. A vertex u is called a *cutvertex* of G if $\{u\}$ is a cutset. We also set $N_G(U) := \{x \in V(G) - U : x$ is adjacent to some vertex in $U\}$, and set $N_G(u) := N_G(\{u\})$; we shall drop the subscript G if there is no danger of confusion. Let H be a graph with $V(H) \subseteq V(G)$. For notational simplicity, we write $G[H]$, G/H, and $G - H$ for $G[V(H)]$, $G/V(H)$, and $G - V(H)$, respectively.

For any two vertices x, y of G, an *x-y path* in G is a path connecting x and y in G. If P is a path, we use $\ell(P)$ to denote the length of P, which is the number of edges of P. For any distinct vertices x, y of a path P, we use $P[x, y]$ to denote the subpath of P between x and y (inclusive), and define

$P[x,y) := P[x,y] - y$, $P(x,y] := P[x,y] - x$, and $P(x,y) := P[x,y] - \{x,y\}$. An edge of G with ends u and v is often denoted by uv, or vu, or $\{u,v\}$. Let S be a family of 2-element subsets of $V(G)$. Then $G+S$ stands for the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. (Note that each edge of G is a 2-element subset of $V(G)$.) If $S = \{\{u_i, v_i\} : i = 1, 2, \ldots, k\}$, then we also write $G + \{u_i v_i : i = 1, 2, \ldots, k\}$ for $G + S$. If $S = \{\{u,v\}\}$, then we set $G + uv := G + S$. Similarly, we can define $G - S$ (with edge set $E(G) - S$).

Given two graphs G and H, by $H \subseteq G$ we mean H is a subgraph of G; the union of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We call H a *minor* of G if there exist disjoint connected sets V_x of G, indexed by $x \in V(H)$, such that, for any distinct $x, y \in V(H)$ with $xy \in E(H)$, there is at least one edge in G with one end in V_x and the other in V_y; we say that the sets V_x, $x \in V(H)$, form a *representation* of H in G, and that G contains an *H-minor* if H is a minor of G. We shall not make effort to distinguish between the edges of H and the edges of G if no ambiguity arises; that is, we may view the edges of H as edges of G.

As usual, $K_{3,t}$ is the complete bipartite graph with one color class having size 3 and the other having size t. In this paper $\tau(G)$ denotes the *maximum number* t such that G contains a $K_{3,t}$-minor.

Let G be a graph and let x, y, z be three distinct vertices of G. We say that a $K_{3,t}$-minor H of G is *rooted at* $\{x,y,z\}$ if H has a representation in G such that $x \in V_1$, $y \in V_2$, $z \in V_3$, where V_1, V_2, V_3 are connected sets of G representing the vertices of H in the color class of size three. Let $\tau(G; x, y, z)$ denote the largest integer t such that G has a $K_{3,t}$-minor rooted at $\{x,y,z\}$. Clearly, $\tau(G; x, y, z) \leq \tau(G)$.

In our proof we shall use rooted $K_{3,s}$-minors to construct $K_{3,t}$-minors, with $s < t$. The following lemma gives a lower bound on $\tau(G; x, y, z)$ in terms of $\tau(G)$, whose proof will be given in Section 3.

Lemma 2.1. *Let G be a 3-connected graph and let x, y, z be three distinct vertices of G. Then*

$$\tau(G; x, y, z) \geq \lceil \tau(G)/3 \rceil.$$

Let H be a subgraph of G. An *H-bridge* of G is a subgraph of G induced by *either* (i) an edge in $E(G) - E(H)$ with both ends in $V(H)$ *or* (ii) the edges in a component D of $G - V(H)$ together with edges of G between D and H. The H-bridges satisfying (ii) are said to be *nontrivial*. If $U \subseteq V(G)$, we may view U as a subgraph of G with vertex set U and no edges. Hence, we shall also speak of U-bridges or bridges of G *associated* with U. If B is a U-bridge, then $V(B) \cap U$ is the set of *attachments* of B. Let X and Y be two disjoint vertex subsets of G (or vertex-disjoint induced subgraphs of G). An $(X \cup Y)$-bridge is called an *(X,Y)-bridge* if its attachment contains at least one vertex in each of X and Y. For instance, any edge between X and Y induces an (X,Y)-bridge, but an edge with both ends in X does not. We speak of (x,Y)-bridge, (X,y)-bridge, and (x,y)-bridge for the cases $X = \{x\}$, $Y = \{y\}$, and $X = \{x\}$ and $Y = \{y\}$, respectively.

A *separation* (K,H) of G consists of two *induced* subgraphs K and H of G such that $K \cup H = G$ and $V(K) - V(H) \neq \emptyset \neq V(H) - V(K)$. (Note that $K \cap H$ may contain edges, which makes this term different from that in the literature.) Clearly, if (K,H) is a separation of G and G is connected, then $S = V(K) \cap V(H)$ is a cutset of G. So we also say that (K,H) is an $|S|$-separation of G. Let xy be an edge of G; a 2-separation (K,H) of G is called *xy-minimal* if xy is an edge of K and there is no other 2-separation (K',H') such that $xy \in K'$, $K' \subseteq K$ and $K' \neq K$.

A *chain of blocks* in a graph G is a sequence $x_0H_0x_1H_1x_2\dots x_mH_mx_{m+1}$ such that each H_i is a block of G, $V(H_i \cap H_{i+1}) = \{x_i\}$ for $1 \le i \le m-1$, $H_i \cap H_j = \emptyset$ whenever $|i-j| \ge 2$, $x_0 \ne x_{m+1}$ when $m = 0$, and if $m \ge 1$ then $x_0 \in V(H_0 - x_1)$ and $x_{m+1} \in V(H_m - x_m)$. We say that this chain of blocks is *from* x_0 *to* x_{m+1}.

The proof of our main theorem is based on graph decompositions. A *3-block* is a 3-connected graph, or a cycle, or a bond (a set of at least three parallel edges sharing two ends). Let us now present an algorithm for decomposing a 2-connected graph into 3-blocks, which is a variant of Tutte's corresponding algorithm. Since bonds play a very limited role in our search for long cycles, they are merged to other 3-blocks whenever possible in the algorithm.

Algorithm 2.2.

Input. *A pair $(H; xy)$, where H is a 2-connected graph and xy is an edge of H.*
Output. *A decomposition of $(H; xy)$ into 3-blocks, a leading block H^*, a set $\Psi(H)$ of virtual edges, and a partial order $\prec$ on $\Psi(H)$.*

Description. *Set $e_0 := xy$. We distinguish among four cases.*

Case 0. *H is a 3-block. In this case, set $H^* := H$ and $\Psi(H) := \{e_0\}$, stop.*

Case 1. *$\{x, y\}$ is a cutset of H. In this case, let $B_1, B_2, \dots, B_m$ be all the nontrivial (x, y)-bridges in H. For $i = 1, 2, \dots, m$, let e_i be a virtual edge between x and y, and let $H_{e_i} := B_i + e_i$; we recursively decompose $(H_{e_i}; e_i)$ into 3-blocks. Set $V(H^*) := \{x, y\}$, $E(H^*) := \{e_0, e_1, \dots, e_m\}$, and $\Psi(H) := \{e_0\} \cup (\bigcup_{i=1}^m \Psi(H_{e_i}))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.*

Case 2. *$\{x, y\}$ is not a cutset of H and $H - e_0$ is a chain of blocks, $x_0H_0x_1H_1x_2\dots x_mH_mx_{m+1}$, with $m \ge 1$, $x_0 = x$ and $y_0 = y$. In this case, for $i = 0, 1, \dots, m$, let $F_i := \{f_i\}$ and $f_i = x_ix_{i+1}$ if x_i and x_{i+1} are adjacent in H and let $F_i = \emptyset$ otherwise, let $B_{i,1}, B_{i,2}, \dots, B_{i,p_i}$ be all the nontrivial (x_i, x_{i+1})-bridges in H_i, let $e_{i,j}$ be a virtual edge between x_i and x_{i+1}, and let $H_{e_{i,j}} := B_{i,j} + e_{i,j}$ for $j = 1, 2, \dots, p_i$; we recursively decompose $(H_{e_{i,j}}; e_{i,j})$ into 3-blocks. Set $V(H^*) := \{x_0, x_1, \dots, x_{m+1}\}$, $E(H^*) := \{e_0\} \cup (\bigcup_{i=0}^m (F_i \cup \{e_{i,1}, e_{i,2}, \dots, e_{i,p_i}\}))$, and $\Psi(H) := \{e_0\} \cup (\bigcup_{i,j} \Psi(H_{e_{i,j}}))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.*

Case 3. *$\{x, y\}$ is not a cutset of H and $H - e_0$ is 2-connected. In this case, let $\{u_i, v_i\}$, for $i = 1, 2, \dots, m$, be all the vertex pairs of H such that there exists an xy-minimal separation (K_i, H_i) of H with $\{u_i, v_i\} = V(K_i) \cap V(H_i)$. For $i = 1, 2, \dots, m$, let $B_{i,1}, B_{i,2}, \dots, B_{i,p_i}$ be all the nontrivial (u_i, v_i)-bridges in H_i, let $e_{i,j}$ be a virtual edge between u_i and v_i for $j = 1, 2, \dots, p_i$, and let $H_{e_{i,j}} := B_{i,j} + e_{i,j}$; we recursively decompose $(H_{e_{i,j}}; e_{i,j})$ into 3-blocks. Set $V(H^*) := \bigcap_{i=1}^m V(K_i)$, $E(H^*) := E(H[V(H^*)]) \cup (\bigcup_{i=1}^m \{e_{i,1}, e_{i,2}, \dots, e_{i,p_i}\})$, and $\Psi(H) := \{e_0\} \cup (\bigcup_{i,j} \Psi(H_{e_{i,j}}))$. Define $g \prec e_0$ for all $g \in \Psi(H) - \{e_0\}$.*

A *multicycle* is obtained from a cycle by adding parallel edges. By definition, both cycles and bonds are multicycles. So in both Case 1 and Case 2, H^* is a multicycle.

As a large portion of our proof will be concerned with graphs obtained from a 3-connected graph by deleting a vertex, let us now apply Algorithm 2.2 to such graphs and exhibit some properties satisfied by its outputs.

Lemma 2.3. *Let G be a 3-connected graph, let x, y, z be three distinct vertices of G with $xz, yz \in E(G)$, and let $H = (G - z) + xy$. Then the outputs of Algorithm 2.2, when applied to $(H; xy)$, satisfy the following properties:*

(i) $\prec$ induces a partial order on $\Psi(H)$;

(ii) H^* is a minor of G;

(iii) H^* is either a multicycle or 3-connected;

(iv) for any virtual edge $f = uv$ in $\Psi(H)$ with $f \neq e_0$, the graph $G_f := G[V(H_f) \cup z] + \{uz, vz\}$ is a 3-connected minor of G. In particular, $\tau(G_f) \leq \tau(G)$.

Proof. (i) Clearly, the relation $\prec$ defined on $\Psi(H)$ satisfies transitivity and antisymmetry. Hence, $\prec$ induces a partial order on $\Psi(H)$, so $(\Psi(H), \prec)$ is a poset.

(ii) If Case 0 or Case 1 occurs then the statement holds trivially. If Case 2 or Case 3 occurs then H^* can be obtained from G by contracting zx to x and contracting $H_e - u$ to v, for each virtual edge $e = uv$ in H^* for which $e \prec xy$ and there is no virtual edge f satisfying $e \prec f \prec xy$. So H^* is a minor of G.

(iii) It is easy to see that H^* is 2-connected. Suppose on the contrary that H^* is neither a multicycle nor 3-connected. Then none of Cases 0-2 described in Algorithm 2.2 occurs and H^* contains a cutset $\{a, b\}$. So $\{a, b\}$ is different from $\{x, y\}$ and is also a cutset in H. Let (A, B) be a separation of H with $V(A) \cap V(B) = \{a, b\}$ and $xy \in A$. Then Case 3 of Algorithm 2.2 guarantees the existence of an xy-minimal separation (K_i, H_i) of H with $K_i \subseteq A$. Let $\{u_i, v_i\} = V(K_i) \cap V(H_i)$. Then $\{a, b\} = \{u_i, v_i\}$, for otherwise a or b would be excluded from H^* by Algorithm 2.2. We thus reach a contradiction because $\{u_i, v_i\}$ is not a cutset of H^*.

(iv) By the construction in Algorithm 2.2, H_f is 2-connected. Since G is 3-connected, every 2-cutset of H_f separates $\{u, v\}$ from some neighbor of z. It follows that G_f is 3-connected. As H is 2-connected, it contains two disjoint paths P_1 and P_2 from $\{x, y\}$ to $\{u, v\}$, where $u \in V(P_1)$ and $v \in V(P_2)$. From Algorithm 2.2 we see that P_i contains no vertex in $V(H_f) - \{u, v\}$ for $i = 1, 2$. So there exist two disjoint connected subgraphs F_1 and F_2 of $H - (V(H_f) - \{u, v\})$ such that $P_i \subseteq F_i$ for $i = 1, 2$ and that $V(F_1) \cup V(F_2) = V(H) - (V(H_f) - \{u, v\})$. If $\{x, y\} = \{u, v\}$ then $xy \in E(G)$; otherwise, since G is 3-connected, there is at least one edge in G between F_1 and F_2. Thus G_f can be obtained from G by contracting F_1 to u and F_2 to v, and hence is a minor of G. ∎

We digress to introduce some important notions before presenting the main result. Let H be a 2-connected graph, let (e, f) be an ordered pair of edges of H, and let A and B be two vertex-disjoint connected subgraphs of H (or two disjoint connected sets of H). We say that the quadruple (A, B, e, f) is a *ladder with top e and bottom f* in H if each of A and B contains precisely one end of each of e and f. For any family F of 2-element subsets of $V(H)$, we use $F \cap [A, B]$ to denote the subfamily of all 2-element subsets in F with one element in A and the other in B.

Let G be a 3-connected graph, let x, y, z be three distinct vertices of G with $xz, yz \in E(G)$, and let $H = (G - z) + xy$. Set $e_0 := xy$, $H_{e_0} := H$, and $G_{e_0} := G$. Suppose we have applied Algorithm 2.2 to $(H; e_0)$. Let us consider an arbitrary virtual edge f in $\Psi(H)$. For each virtual edge e in $\Psi(H_f)$, from Lemma 2.3(d) (with G_e and G_f in place of G_f and G over there, respectively) it follows that

$\tau(G_e) \le \tau(G_f)$. We call e *full* with respect to f if $\tau(G_e) = \tau(G_f)$, and set

$$\Psi_1(H_f) := \{e \in \Psi(H_f) : e \text{ is full with respect to } f\} \quad \text{and} \quad \Psi_2(H_f) := \Psi(H_f) - \Psi_1(H_f).$$

For each $e \in \Psi(H_f)-\{f\}$, let $\theta(e, H_f)$ denote the maximum size of an anti-chain X (recall Lemma 2.3(a)) in $\Psi_1(H_f) \cap [A, B]$, taken over all ladders (A, B, e, f) with top e and bottom f in $H_f + e$, such that

- if $e \in \Psi_1(H_f)$ then $e \in X$, and
- if $e \in \Psi_2(H_f)$ (so $e \notin X$) then no element of X is comparable with e.

Notice that $X \cup \{e\}$ is always an antichain.

For each $e \in \Psi(H_f)-\{f\}$, let $\phi(e, H_f)$ denote the maximum size of an anti-chain Y in $\Psi(H_f)\cap[A, B]$, taken over all ladders (A, B, e, f) with top e and bottom f in $H_f + e$, such that

- $e \in Y$, and
- $|Y \cap \Psi_1(H_f)| = \theta(e, H_f)$.

Since $Y \cap \Psi_1(H_f)$ gives an antichain realizing $\theta(e, H_f)$, we obtain $\theta(e, H_f) \le \phi(e, H_f)$.

Set $\theta(f, H_f) = \phi(f, H_f) := 0$, and set

$$\theta(H_f) := \max_{e\in\Psi(H_f)} \{\theta(e, H_f)\} \quad \text{and} \quad \phi(H_f) := \max_{\substack{e\in\Psi(H_f)\\ \theta(e,H_f)=\theta(H_f)}} \{\phi(e, H_f)\}. \tag{2.1}$$

By definition, $\theta(H_f) \le \phi(H_f)$. Moreover, $\theta(H_f) = \phi(H_f) = 0$ if H_f is 3-connected (in this case $\Psi(H_f) = \Psi_1(H_f) = \{f\}$, so $X = Y = \emptyset$). For any positive integer t, set

$$\delta(t, H_f) := \frac{1}{3^{\theta(H_f)}} \left(1 - \frac{\phi(H_f) - \theta(H_f)}{3t}\right). \tag{2.2}$$

As $H = H_{e_0}$, we have $\theta(H) = \theta(H_{e_0})$, $\phi(H) = \phi(H_{e_0})$, and $\delta(t, H) = \delta(t, H_{e_0})$.

The following observations aim to give a good estimate of the parameter $\delta(t, H)$.

Lemma 2.4. *Let G be a 3-connected graph, let x, y, z be three distinct vertices of G with $xz, yz \in E(G)$, and let $H = (G - z) + xy$. Suppose Algorithm 2.2 has been applied to $(H; xy)$. Let $f = uv$ (possibly $f = e_0 := xy$) be a virtual edge in $\Psi(H)$, and let $G_f := G[V(H_f) \cup z] + \{uz, vz\}$. Then the following statements hold:*

(i) $\theta(H_f) \le \phi(H_f) \le \tau(G_f)$;

(ii) $\theta(H_f) \le 3$ *and equality holds only if* $\phi(H_f) = 3$;

(iii) *for any* $t \ge \tau(G_f)$, *we have* $1/27 \le \delta(t, H_f) \le 1$, *and* $\delta(t, H_f) \le 1/3$ *if* $\theta(H_f) \ge 1$;

(iv) *if* $\tau(G_f) = \tau(G)$, *then* $\delta(t, H) \le \delta(t, H_f)$ *for any* $t \ge \tau(G)$.

Proof. (i) From the definition it follows instantly that $\theta(H_f) \le \phi(H_f)$. To prove that $\phi(H_f) \le \tau(G_f)$, we may assume $\phi(H_f) > 0$. Thus the definition guarantees the existence of a virtual edge e in $\Psi(H_f)$, a ladder (A, B, e, f) with top e and bottom f in $H_f + e$, and an anti-chain Y in $\Psi(H_f) \cap [A, B]$ such that $e \in Y$, $|Y \cap \Psi_1(H_f)| = \theta(H_f)$, and $|Y| = \phi(H_f)$. For each $g \in Y$, set $V_g = V(H_g) - (A \cup B)$. Then the sets $A, B, \{z\}$ and V_g (for all $g \in Y$) form a representation of a $K_{3,|Y|}$-minor of G_f rooted at $\{u, v, z\}$. So $\phi(H_f) = |Y| \le \tau(G_f)$.

(ii) We only need to consider the case when $\theta(H_f) > 0$. Let e be a virtual edge in $\Psi(H_f)$, (A, B, e, f) a ladder with top e and bottom f in $H_f + e$, and Y an anti-chain in $\Psi(H_f) \cap [A, B]$ that $e \in Y$, $|Y \cap \Psi_1(H_f)| = \theta(H_f)$, and $|Y| = \phi(H_f)$. Let $X := Y \cap \Psi_1(H_f)$. Then $|X| = \theta(H_f)$. Suppose $\tau(G_f) = q$. For each $g \in Y$, let $g = u_g v_g$. By Lemma 2.3, G_g is a 3-connected minor of G_f. Observe that for each $g \in X$, we have $g \in \Psi_1(H_f)$, so $\tau(G_g) = \tau(G_f) = q$. From Lemma 2.1 we deduce that G_g has a $K_{3,\lceil q/3 \rceil}$-minor Σ_g rooted at $\{u_g, v_g, z\}$. Clearly, for each $g \in Y - X$, G_g has a $K_{3,1}$-minor Σ_g rooted at $\{u_g, v_g, z\}$. Let G' denote the graph $G[A] \cup G[B] \cup (\bigcup_{g \in Y} \Sigma_g)$. Then G' contains a $K_{3,p}$-minor of G (this can be seen by contracting A to a single vertex and B to another vertex), with $p \geq \lceil q/3 \rceil |X| + (|Y| - |X|)$. Since $p \leq q$ and $|X| \leq |Y|$, we have $q \geq (q/3)|X|$ (so $|X| \leq 3$), and equality holds only if $|Y| = |X|$ and q is a multiple of 3.

(iii) By (i), we have $\theta(H_f) \leq \phi(H_f) \leq \tau(G_f) \leq t$. So $0 \leq \frac{\phi(H_f) - \theta(H_f)}{3t} \leq \frac{1}{3}$ and hence $\delta(t, H_f) \geq \frac{1}{3^{\theta(H_f)}} \frac{2}{3}$. If $\theta(H_f) \leq 2$, then $\delta(t, H_f) \geq \frac{2}{27}$. If $\theta(H_f) = 3$, then $\phi(H_f) = 3$ by (ii) and hence, by definition, $\delta(t, H_f) = \frac{1}{27}$. In view of (ii), $\theta(H_f) \leq 3$, so the inequality $\delta(t, H_f) \geq \frac{1}{27}$ always holds. As $\delta(t, H_f) \leq \frac{1}{3^{\theta(H_f)}}$, the upper bound follows instantly.

(iv) We may assume that $f \neq e_0 = xy$, for otherwise $H_f = H$, so the statement holds trivially. Since $\tau(G_f) = \tau(G)$, we have $\Psi_1(H_f) \subseteq \Psi_1(H_{e_0})$; and hence $\theta(H_f) \leq \theta(H)$ (by (2.1)). If $\theta(H_f) = \theta(H)$ then, by (2.1), we have $\phi(H_f) \leq \phi(H)$; and so $\delta(t, H) \leq \delta(t, H_f)$ (by (2.2)). If $\theta(H_f) \leq \theta(H) - 1$, then

$$\delta(t, H_f) = \frac{1}{3^{\theta(H_f)}}(1 - \frac{\phi(H_f) - \theta(H_f)}{3t}) \geq \frac{1}{3^{\theta(H_f)}}(1 - \frac{t}{3t}) = \frac{1}{3^{\theta(H_f)}}\frac{2}{3} \geq \frac{2}{3^{\theta(H)}} \geq \delta(t, H),$$

where the first inequality follows from (i). So the statement is established in either case. ∎

Now we are ready to state the main result of this paper, which implies Theorem 1.3 immediately.

Theorem 2.5. *Let $t \geq 1$ be an integer, let $G = (V, E)$ be a 3-connected graph with $\tau(G) \leq t$, let $\alpha(t) := (1/2)^{t(t-1)}$, and let $\beta := \log_b 2$, where $b = 1729$. Then the following statements hold:*

(*a*) *For any distinct $x, y, z \in V$ with $xz,\ yz \in E$, there exists an x-y path in $G - z$ of length at least $\alpha(t)\,(\delta(t, H)(|G| - 1))^{\beta}$, where $H = (G - z) + xy$ and $\delta(t, H)$ is as defined in (2.2).*

(*b*) *For any distinct $e, f = xy \in E$, there exists an x-y path in G through e of length at least $\alpha(t)\left(\frac{|G|}{28}\right)^{\beta} + 1$.*

(*c*) *For any $xy \in E$, there exists an x-y path in G of length at least $\alpha(t)|G|^{\beta}$.*

Outline of Proof. Let $n := |G|$. We prove by double induction on n and t. Obviously G contains a path as specified in each of (a), (b) and (c) with length at least 2. If $n \leq b^{t(t-1)}$ then $\alpha(t) n^{\beta} \leq 1$. So the lower bounds specified in (a)-(c) are all at most 2 for $\delta(t, H) \leq 1$, and hence (a)-(c) hold simultaneously in this case. If $\tau(G) = 1$ then G is K_4 (the complete graph on four vertices). Thus (a)-(c) all hold trivially again. Therefore, we proceed to the induction step and assume that $t \geq 2$, $n > b^{t(t-1)}$, and statements (a)-(c) have been established for all graphs with at most $n-1$ vertices and for all graphs with no $K_{3,t}$-minors. The inductive processes of statements (a)-(c) will take up the last three sections of this paper.

We point out that the proofs of statements (a) and (c) are substantially different from their counterparts in [4].

3 Rooted $K_{3,t}$-Minors and the Function $x^{\log_b 2}$

The purpose of this section is to give a proof of Lemma 2.1, and state several lemmas concerning the function the function $x^{\log_b 2}$, which will be used repeatedly in the proof of Theorem 2.5.

Before proving Lemma 2.1, we remark that the bound in Lemma 2.1 is sharp. To see this, let $G = (A \cup B, E)$ be a graph such that

- $A = \{a_1, a_2, \ldots, a_5\}$ and $B = \{b_1, b_2, \ldots, b_{3t}\}$ for any positive integer $t \geq 3$;
- a_1, a_2, a_3 are pairwise adjacent;
- a_i is adjacent to vertices $b_{(i-1)t+1}, b_{(i-1)t+2}, \ldots, b_{it}$ for $i = 1, 2, 3$;
- a_j is adjacent to vertices $b_1, b_2, \ldots, b_{3t}$ for $j = 4, 5$.

Clearly, G is 3-connected and contains a $K_{3,3t}$-minor. However, it is easy to verify that G contains no $K_{3,t+1}$-minor rooted at $\{a_1, a_2, a_3\}$.

In our proof of Lemma 2.1 we shall use contractible edges. An edge $e = uv$ in a 3-connected graph G is called *contractible* if G/e is also 3-connected and *noncontractible* otherwise. Obviously, $e = uv$ is noncontractible in G if and only if $\{u, v\}$ is contained in a 3-cutset of G. Let $E_c(G)$ (resp. $E_n(G)$) denote the set of contractible (resp. noncontractible) edges of G, and let $\mathcal{N}(G)$ denote the collection of all triples (e, S_e, C_e), where $e \in E_n(G)$, S_e is a 3-cutset of G containing $V(e)$, and C_e is a component of $G - S_e$. We call (e, S_e, C_e) *minimal* if there exists no $(f, S_f, C_f) \in \mathcal{N}(G)$ such that C_f is a proper subgraph of C_e.

Lemma 3.1. *Let G be a 3-connected graph and let $(e, S_e, C_e) \in \mathcal{N}(G)$ be minimal. Then all edges of C_e and all edges from C_e to $S_e - V(e)$ are contractible in G.*

Proof. Let f be an edge of C_e or an edge from C_e to $S_e - V(e)$. If f is noncontractible, then $V(f)$ is contained in a 3-cutset S_f of G. It is thus a routine matter to check that $S_f \subseteq S_e \cup V(C_e)$. Consequently, some component of $G - S_f$ is properly contained in C_e, a contradiction. ■

By using similar arguments, Ando *et al.* [1] obtained the following result.

Lemma 3.2. *Let G be a 3-connected graph and let v be a vertex of G with degree 3. Then*

(i) G has a contractible edge incident with v, and

(ii) if there is exactly one contractible edge incident with v, then the noncontractible edges incident with v induce a triangle T whose vertices all have degree 3 in G. ■

Observe that the triangle T specified in (ii) is contractible; that is, G/T is 3-connected.

Proof of Lemma 2.1. Let $\tau(G) = t$ and let $V_1, V_2, \ldots, V_{t+3}$ denote a representation of a $K_{3,t}$-minor in G with color classes $\{V_1, V_2, V_3\}$ and $\{V_4, V_5, \ldots, V_{t+3}\}$. Clearly, we may assume that $\bigcup_{i=1}^{t+3} V_i = V(G)$. Since G is 3-connected, it has a $K_{3,1}$-minor rooted at $\{x, y, z\}$. Thus the statement holds for $t \leq 3$. It remains to consider the case when $t \geq 4$.

Since the statement is trivial if $|V_s| = 1$ for $s = 1, 2, \ldots, t+3$, we may assume that $|V_s| \geq 2$ for some s with $1 \leq s \leq t+3$, and that the assertion has been established for smaller graphs.

(1) We may choose $V_1, V_2, \ldots, V_{t+3}$ so that for every $1 \leq s \leq t+3$, if $|V_s| \geq 2$ then $|V_s \cap \{x, y, z\}| \geq 2$.

To justify this, let us assume that $|V_s| \geq 2$ while $|V_s \cap \{x, y, z\}| \leq 1$ for some subscript s with $1 \leq s \leq t+3$. Then no edge e in $G[V_s]$ is contractible, for otherwise, G/e is 3-connected and $\tau(G/e) = t$ (as both ends of e are in V_s). It follows from induction hypothesis that $\tau(G; x, y, z) \geq \tau(G/e; x, y, z) \geq t/3$, we are done. Hence, there exists $(e, S_e, C_e) \in \mathcal{N}(G)$. Let $\{a\} = S_e - V(e)$. Then $a \in V_r$ for some r (possibly $r = s$). By the structure of $K_{3,t}$-minor, $G - (V_s \cup V_r)$ is connected. Hence we can choose (e, S_e, C_e) so that $V(C_e) \subseteq V_s \cup V_r$. For technical reasons, we further assume that $V_1, V_2, \dots, V_{t+3}$ and (e, S_e, C_e) are chosen so that $|C_e|$ is maximized.

Let $V'_s = V_s \cup V(C_e)$, $V'_r = V_r - V(C_e)$, and $V'_i = V_i$ for all $i \neq r, s$. Then $V'_1, V'_2, \dots, V'_{t+3}$ form a representation of a $K_{3,t}$-minor.

We propose to show that $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \geq 2$. Otherwise, $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \leq 1$. Choose minimal $(f, S_f, C_f) \in \mathcal{N}(G)$ such that $S_f \cup V(C_f) \subseteq S_e \cup V(C_e)$ and $V(C_f) \subseteq V(C_e)$. Let $f = uv$ and $S_f = \{u, v, w\}$. Suppose $a \neq w$ (so $w \in V(C - e) \cup V(e)$). By Lemma 3.1, any edge ww' with $w' \in V(C_f)$ is contractible in G. Since G/ww' has a $K_{3,t}$-minor, we have $w, w' \in \{x, y, z\}$, for otherwise the present lemma follows from induction. Thus we may assume $a = w$. Then $u, v \in V'_s$. If $|V(C_f)| \geq 2$ then again by Lemma 3.1 any edge g in C_f is contractible, and so we may assume $V(g) \subset \{x, y, z\}$ as before. Hence there is only one vertex b in C_f. It follows that $d(b) = 3$ and bu, $bv \in E(G)$. If bu or $bv \in E_c(G)$, then again we can either apply induction to show that $\tau(G; x, y, z) \geq t/3$ or conclude that $b, u \in \{x, y, z\}$ or $b, v \in \{x, y, z\}$. Therefore we may assume that bu, $bv \in E_n(G)$. By Lemma 3.2 and the remark on (ii) of Lemma 3.2, $d(u) = d(v) = 3$ and the triangle $T = buvb$ is contractible. Note that $b, u, v \in V'_s$. If $|\{b, u, v\} \cap \{x, y, z\}| \leq 1$ then the lemma follows from induction. So we may assume the contrary, which implies that $|(V(C_e) \cup V(e)) \cap \{x, y, z\}| \geq 2$, as desired.

From the assumption on V_s we deduce that $r \neq s$. Next, let us show that $|V'_j| = 1$ for any $j \neq s$ (and hence (1) follows). Suppose $|V'_j| \geq 2$ for some $j \neq s$. Then $|V'_j \cap \{x, y, z\}| \leq 1$ as $|V'_s \cap \{x, y, z\}| \geq 2$. Using the same argument with respect to $V'_1, V'_2, \dots, V'_{t+3}$ (in place of $V_1, V_2, \dots, V_{t+3}$) and j (in place of s), we deduce that no edge e' in $G[V'_j]$ is contractible, so there exists $(e', S'_e, C'_e) \in \mathcal{N}(G)$ such that $V(C'_e) \subseteq V'_j \cup V'_k$ for some k and $G - (V'_j \cup V'_k)$ is connected. As before, $|(V(C'_e) \cup V(e')) \cap \{x, y, z\}| \geq 2$. This implies that $V(C'_e) \cap V(C_e) \cap \{x, y, z\} \neq \emptyset$. Since $V(e') \cap V(e) = \emptyset$, $V(e')$ can have at most one vertex in common with $S_e \cup C_e$. By 3-connectedness of G, we thus have $S'_e \subseteq G - V(C_e)$. It follows that C_e is properly contained in C'_e. So $V'_1, V'_2, \dots, V'_{t+3}$ and (e', S'_e, C'_e) contradict the choices of $V_1, V_2, \dots, V_{t+3}$ and (e, S_e, C_e), completing the proof of (1).

By (1), there exists a unique subscript s such that $|V_s| \geq 2$. Renaming vertices if necessary, we assume that $x, y \in V_s$. Let v_i be the only vertex in V_i for all $i \neq s$. Since G is 3-connected, there exist three disjoint paths P_x, P_y, P_z from x, y, z to some distinct vertices v_i, v_j, v_k, respectively, which are disjoint from v_ℓ for all $\ell \notin \{i, j, k, s\}$, where $s \notin \{i, j, k\}$. Now let us consider two possible cases.

Case 1. $z \in V_s$.

Since V_s is connected, it can be partitioned into three connected sets Q_x, Q_y, Q_z such that $V(P_x) \subseteq Q_x$, $V(P_y) \subseteq Q_y$, and $V(P_z) \subseteq Q_z$.

Subcase 1.1. $s \geq 4$.

Without loss of generality, we assume that $s = 4$. If $\max\{i, j, k\} = 3$ then $Q_x \cup V_i, Q_y \cup V_j, Q_z \cup V_k, V_5, \dots, V_{t+3}$ form a representation of $K_{3,t-1}$-minor in G rooted at $\{x, y, z\}$, so $\tau(G; x, y, z) \geq t-1 \geq t/3$ as $t \geq 4$.

Suppose $\min\{i,j,k\} \geq 5$. Then we may assume $i = 5, j = 6, k = 7$. Since $Q_x \cup V_5, Q_y \cup V_6, Q_z \cup V_7, V_1, V_2, V_3$ form a representation of a $K_{3,3}$-minor in G rooted at $\{x,y,z\}$, we may assume that $t \geq 10$. Thus $Q_x \cup V_5 \cup V_1, Q_y \cup V_6 \cup V_2, Q_z \cup V_7 \cup V_3, V_8, \ldots, V_{t+3}$ form a representation of a $K_{3,t-4}$-minor in G rooted at $\{x,y,z\}$. Hence $\tau(G;x,y,z) \geq t-4 \geq t/3$ as $t \geq 10$.

So we suppose $\min\{i,j,k\} \leq 3$. Renaming subscripts if necessary, we assume $i = 1$. If $\min\{j,k\} \leq 3$ then we may assume $j = 2$ and $k = 5$; in this case, $Q_x \cup V_1, Q_y \cup V_2, Q_z \cup V_5 \cup V_3, V_6, \ldots, V_{t+3}$ form a representation of a $K_{3,t-2}$-minor in G rooted at $\{x,y,z\}$. So $\tau(G;x,y,z) \geq t-2 \geq t/3$ as $t \geq 4$. Hence we assume $j = 5$ and $k = 6$. Note that $Q_x \cup V_1 \cup V_7, Q_y \cup V_5, Q_z \cup V_6, V_2, V_3$ form a representation of a $K_{3,2}$-minor in G rooted at $\{x,y,z\}$; so we may assume $t \geq 7$. Since $Q_x \cup V_1, Q_y \cup V_5 \cup V_2, Q_z \cup V_6 \cup V_3, V_7, \ldots, V_{t+3}$ form a representation of a $K_{3,t-3}$-minor in G rooted at $\{x,y,z\}$, we have $\tau(G;x,y,z) \geq t-3 \geq t/3$ as $t \geq 7$.

Subcase 1.2. $s \leq 3$.

Clearly we may assume that $s = 1$. By the pigeonhole principle, there exists $\{i_1, i_2, \ldots, i_p\} \subseteq \{4, 5, \ldots, t+3\}$, with $p = \lceil t/3 \rceil$, such that one of the following (a), (b), and (c) holds:

(a) $j, k \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in Q_x;

(b) $i, k \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in Q_y; and

(c) $i, j \notin \{i_1, i_2, \ldots, i_p\}$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ all have neighbors in Q_z.

Since $t \geq 4$, we have $t - p \geq 2$.

If $\{j,k\} \subseteq \{v_4, v_5, \ldots, v_{t+3}\}$, then symmetry allows us to assume that (a) occurs. Thus $Q_x, Q_y \cup V_j \cup V_2, Q_z \cup V_k \cup V_3, V_{i_1}, \ldots, V_{i_p}$ form a representation of a $K_{3,p}$-minor in G rooted at $\{x,y,z\}$. Now suppose $\{j,k\} = \{2,3\}$. If (a) occurs then $Q_x, Q_y \cup V_j, Q_z \cup V_k, V_{i_1}, \ldots, V_{i_p}$ form a representation of a $K_{3,p}$-minor in G rooted at $\{x,y,z\}$. If (b) occurs then $Q_x \cup V_i \cup V_j, Q_y, Q_z \cup V_k, V_{i_1}, \ldots, V_{i_p}$ form a representation of a $K_{3,p}$-minor in G rooted at $\{x,y,z\}$. The proof goes along the same line if (c) occurs.

It remains to consider that $j \in \{2,3\}$ and $k \geq 4$. If (b) occurs then $Q_x \cup V_i \cup V_2, Q_y, Q_z \cup V_k \cup V_3, V_{i_1}, \ldots, V_{i_p}$ form a representation of a $K_{3,p}$-minor in G rooted at $\{x,y,z\}$. By symmetry we may assume that (a) occurs and $j = 2$. Then $Q_x, Q_y \cup V_2, Q_z \cup V_k \cup V_3, V_{i_1}, \ldots, V_{i_p}$ form a representation of a $K_{3,p}$-minor in G rooted at $\{x,y,z\}$.

Case 2. $z = v_r$ for some $r \neq s$.

In this case we may assume that $P_z = z = v_k$. So $r = k$. Let us partition V_s into two connected sets Q_x, Q_y such that $V(P_x) \subseteq Q_x$ and $V(P_y) \subseteq Q_y$.

Subcase 2.1. $s \leq 3$.

Without loss of generality, we may assume $s = 1$. By the pigeonhole principle and by symmetry, we may assume that $v_4, v_5, \ldots, v_p$ all have neighbors in Q_x, such that $j, k \notin \{4, 5, \ldots, p\}$, $p - 3 \geq (t-2)/2$, and $p - 3 \geq (t-1)/2$ if $\min\{j,k\} \leq 3$.

If $k \leq 3$, say $k = 3$, then $Q_x, Q_y \cup V_j \cup V_2, V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in G rooted at $\{x,y,z\}$. Hence $\tau(G;x,y,z) \geq (t-1)/2 \geq t/3$ (since $t \geq 4$). Similarly, if $j \leq 3$, say $j = 3$, then $Q_x, Q_y \cup V_j, V_2 \cup V_k, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in G rooted at $\{x,y,z\}$; and hence $\tau(G;x,y,z) \geq t/3$.

So we may assume $\min\{j,k\} > p$. Then $Q_x \cup V_4, Q_y \cup V_j, V_k, V_2, V_3$ form a representation of a $K_{3,2}$-minor in G rooted at $\{x,y,z\}$; and hence we may assume $t \geq 7$. Since $Q_x, Q_y \cup V_j \cup V_2, V_k \cup V_3, V_4, \ldots, V_p$ form a representation of a $K_{3,p-3}$-minor in G rooted at $\{x,y,z\}$, we have $\tau(G;x,y,z) \geq (t-2)/2 \geq t/3$

(for $t \geq 7$).

Subcase 2.2. $s \geq 4$.

We may assume $s = 4$. Without loss of generality, we may assume that $i = 1$ if $i \leq 3$, $j = 2$ if $j \leq 3$, and $k = 3$ if $k \leq 3$.

Suppose $\min\{i, j, k\} > 3$. Then we may assume $t \geq 10$, for $Q_x \cup V_i, Q_y \cup V_j, V_k, V_1, V_2, V_3$ form a representation of a $K_{3,3}$-minor in G rooted at $\{x, y, z\}$. Since $Q_x \cup V_i \cup V_1, Q_y \cup V_j \cup V_2, V_k \cup V_3$ and $\{V_5, V_6, \ldots, V_{t+3}\} - \{V_i, V_j, V_k\}$ form a representation of a $K_{3,t-4}$-minor in G rooted at $\{x, y, z\}$, we have $\tau(G; x, y, z) \geq t - 4 \geq t/3$.

If $k \leq 3$, then $k = 3$. Since V_4 is adjacent to both V_1 and V_2, the definition of i, j and symmetry allow us to assume that $i = 1$ or $j = 2$, say the former. It follows that $t \geq 7$ because $Q_x \cup V_1, Q_y \cup V_j \cup V_2, V_3$ and $\{V_5, V_6, V_7\} - \{V_j\}$ form a representation of $K_{3,p}$-minor in G rooted at $\{x, y, z\}$, with $p \geq 2$. Since $Q_x \cup V_1, Q_y \cup V_j \cup V_2, V_3$ and $\{V_5, V_6, \ldots, V_{t+3}\} - \{V_j\}$ form a representation of $K_{3,q}$-minor in G rooted at $\{x, y, z\}$, with $q \geq t - 3$, we have $\tau(G; x, y, z) \geq t - 3 \geq t/3$.

So we may assume $k \geq 5$. If $i, j \leq 3$ then $i = 1$, $j = 2$, and $Q_x \cup V_1, Q_y \cup V_2, V_k \cup V_3$ and $\{V_5, V_6, \ldots, V_{t+3}\} - \{V_k\}$ form a representation of a $K_{3,t-2}$-minor in G rooted at x, y, z. Thus $\tau(G; x, y, z) \geq t - 2 \geq t/3$ (as $t \geq 4$). So we may assume by symmetry that $i = 1$ and $j \geq 5$. Then we may assume $t \geq 7$ since there exists $\ell \in \{5, 6, \ldots, t+3\} - \{j, k\}$ such that $Q_x \cup V_1 \cup V_\ell, Q_y \cup V_j, V_k, V_2, V_3$ form a representation of a $K_{3,2}$-minor in G rooted at $\{x, y, z\}$. As $Q_x \cup V_1, Q_y \cup V_j \cup V_2, V_k \cup V_3$ and $\{V_5, V_6, \ldots, V_{t+3}\} - \{V_j, V_k\}$ form a representation of a $K_{3,t-3}$-minor in G rooted at $\{x, y, z\}$, we have $\tau(G; x, y, z) \geq t - 3 \geq t/3$ (for $t \geq 7$). This completes the proof of our lemma. ■

To ensure 3-connectedness of some graph minors involved in our proof, we shall appeal to the following lemma, which was first established in [4].

Lemma 3.3. *Let G be a 3-connected graph, and let H be an induced 2-connected subgraph of G such that $U := G - V(H)$ is connected. Then G/U is 3-connected.* ■

The following property of the function $f(x) = x^{\log_b 2}$ allows us to discard some parts of the input graph in our search procedure; see [4] for its proof.

Lemma 3.4. *For any integer $b \geq 4$ and any $m \geq n > 0$,*

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + (b - 1)n)^{\log_b 2}.$$ ■

Corollary 3.5. *Let $a \geq 1$ and $b \geq 4$ be integers, and let $m > 0$ and $n > 0$. If $m \geq \frac{n}{a}$, then*

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + \frac{b-1}{a} n)^{\log_b 2}.$$ ■

Repeated application of Corollary 3.5 yields the following statement.

Corollary 3.6. *Suppose $m, n_1, \ldots, n_k$ are positive numbers such that $m \geq \frac{n_i}{a}$ for $1 \leq i \leq k$. Then, for any integer $b \geq 4$,*

$$m^{\log_b 2} + \sum_{i=1}^{k} n_i^{\log_b 2} \geq (m + \frac{b-1}{a} \sum_{i=1}^{k} n_i)^{\log_b 2}.$$ ■

4 Cycles in Weighted Graphs

In our proof we shall use weights to keep track of the lengths of paths generated in 3-blocks output by Algorithm 2.2 and some of their unions, so we study the longest cycle problem on weighted graphs (with parallel edges allowed) in this section.

Let H be a 2-connected graph, let $S \subseteq E(H)$, and let (e, f) be an ordered pair of distinct edges in H. For each ladder $L = (A, B, e, f)$ with top e and bottom f in H, the edges in $S \cap [A, B] - \{e\}$ are called the *S-rungs* of L. Note that the bottom f is counted as an S-rung whenever $f \in S$ while the top e will never be counted. Moreover, S may contain parallel edges.

Let $f = xy \in E(H)$ and let P be an x-y path in H. For any $e = uv \in E(P)$ with x, u, v, y on P in this order, a *ladder generated by P with top e* is a ladder (A, B, e, f) with $P[x, u] \subseteq A$ and $P[v, y] \subseteq B$. Let $\sigma_{H,S}(P, e)$, or $\sigma(P, e)$ (if there is no confusion), denote the maximum number of S-rungs of a ladder generated by P with top e. In the extreme case $E(P) = \{f\}$, we define $\sigma(P, f)$ as 1 if $f \in S$ and $|S| \geq 2$ and as 0 otherwise. (The theorem and its corollary established in this section will only be used in Section 5, where we always have $f \notin S$.)

The following is a strengthening of Theorem 3.1 in [4].

Theorem 4.1. *Let H be a 2-connected graph, let $\omega : E(H) \mapsto \mathbb{R}^+$, and let $S = \{e \in E(H) : \omega(e) > 0\}$. Then for any $xy \in E(H)$, there exists an x-y path P in H such that*

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \omega(H),$$

where $\omega(H) = \sum_{e \in E(H)} \omega(e)$.

Proof. Note that $\omega(H) = \omega(S) := \sum_{e \in S} w(e)$. We proceed by induction on $|E(H)| + |S|$. If $|S| = 0$, then $\omega(H) = 0$. Hence any x-y path P in G is as desired. If $|S| = 1$, then H has an x-y path P containing the edge in S for H is 2-connected. Clearly, $\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \omega(H)$. So we may assume $|S| \geq 2$.

Suppose $|E(H)| = 3$. Then H is a triangle. Let P and Q be the two x-y paths in H, with $\omega(P) \geq \omega(Q)$. If $S \subseteq E(P)$, then

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \sum_{e \in E(P)} \omega(e) = \omega(H).$$

If $S \cap E(Q) \neq \emptyset$, then $\sigma(P, e) \geq 1$ for any $e \in E(P)$. It follows that

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq 2\omega(P) \geq \omega(H).$$

So the desired statement holds in either case. Therefore we assume hereafter that $|E(H)| \geq 4$. The remainder of the proof is divided into two cases.

Case 1. $\{x, y\}$ is a cutset of H or S contains an edge incident with both x and y.

In this case there exist subgraphs H_1 and H_2 of H such that $H_1 \cup H_2 = H$, $V(H_1) \cap V(H_2) = \{x, y\}$, $E(H_1) \cap E(H_2) = \emptyset$, and for each i either $|H_i| \geq 3$ or $E(H_i) \cap S \neq \emptyset$. Renaming subscripts if necessary, we assume $\omega(H_1) \geq \omega(H_2)$, which implies $2\omega(H_1) \geq \omega(H)$.

Suppose H_1 is induced by an edge $f \in S$. Then f is incident with both x and y. Let $P = H_1$. As $|S| \geq 2$, by definition $\sigma(P, f) = 1$. Thus $\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq 2\omega(f) = 2\omega(H_1) \geq \omega(H)$.

So we assume that $|H_1| \geq 3$. Let $H^* = H_1$ if H_1 contains an edge between xy; otherwise let $H^* := H_1 + xy$. Let $S^* := S \cap E(H_1)$, and let $\omega^* : E(H^*) \mapsto \mathbb{R}^+$ be such that $\omega^*(e) = \omega(e)$ if $e \in E(H_1)$ and $\omega^*(xy) = 0$ if $xy \notin E(H_1)$. Then $S^* := \{e \in E(H^*) : w^*(e) > 0\}$. Note that H^* is 2-connected and $|E(H^*)| + |S^*| < |E(H)| + |S|$. So the induction hypothesis on (H^*, ω^*) guarantees the existence of an x-y path P in H^* such that

$$\sum_{e \in E(P)} 2^{\sigma^*(P,e)} \omega^*(e) \geq \omega^*(H^*) = \omega(S^*) = \omega(H_1),$$

where $\sigma^*(P, e)$ is defined for P in H^*.

If $S \cap E(H_2) = \emptyset$, then $\omega(H_1) = \omega(H)$ and $\sigma(P, e) = \sigma^*(P, e)$ for all $e \in E(P)$. Thus

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) = \sum_{e \in E(P)} 2^{\sigma^*(P,e)} \omega^*(e) \geq \omega(H_1) = \omega(H).$$

So we may assume that $S \cap E(H_2) \neq \emptyset$. Using 2-connectedness of H, we have $\sigma(P, e) \geq \sigma^*(P, e) + 1$ for all $e \in E(P)$. Hence

$$\sum_{e \in E(P)} 2^{\sigma(P,e)} \omega(e) \geq \sum_{e \in E(P)} 2^{\sigma^*(P,e)+1} \omega^*(e) \geq 2\omega(H_1) \geq \omega(H).$$

Case 2. $\{x, y\}$ is not a cutset of H and no edge in S is incident with both x and y.

In this case y is contained in a unique block of $H - x$, denoted by Y. Let X be an (x, Y)-bridge of H with $\omega(X)$ maximum, and let u be the unique vertex in $V(X) \cap V(Y)$. If X is a nontrivial (x, Y)-bridge of H, then $u \neq y$ because $\{x, u\}$ is a cutset of H while $\{x, y\}$ is not. Otherwise, we may choose X so that $u \neq y$, since no edge in S is between x and y. Thus we can assume that $u \neq y$.

Let $S_X = S \cap E(X)$ and $S_Y = S \cap E(Y)$. Clearly, both $|E(X)| + |S_X|$ and $|E(Y)| + |S_Y|$ are less than $|E(H)| + |S|$. Let ω_X and ω_Y be the restrictions of ω on X and Y, respectively. Without loss of generality, we may assume that xu is an edge in X, for otherwise we add such a dummy edge to X and define $\omega(xu) = 0$. Similarly, we assume that yu is an edge in Y.

If $|X| = 2$, set $P_x := X$. If $|X| \geq 3$, applying the induction hypothesis on (X, ω_X), we find an x-u path P_x (excluding the dummy edge, if any) in X such that $\sum_{e \in E(P_x)} 2^{\sigma_X(P_x,e)} \omega_X(e) \geq \omega_X(X) = \omega(X)$, where $\sigma_X(P_x, e)$ is the maximum number of S_X-rungs in a ladder induced by P_x in X with top e.

If $|Y| = 2$, set $P_y := Y$. If $|Y| \geq 3$, applying the induction hypothesis on (Y, ω_Y), we find a u-y path P_y (excluding the dummy edge, if any) in Y such that $\sum_{e \in E(P_y)} 2^{\sigma_Y(P_y,e)} \omega_Y(e) \geq \omega_Y(Y) = \omega(Y)$, where $\sigma_Y(P_y, e)$ is the maximum number of S_Y-rungs in a ladder induced by P_y in Y with top e.

Let $P := P_x \cup P_y$. Clearly, $\sigma(P, e) \geq \sigma_Y(P_y, e)$ for any $e \in E(P_y)$. Let k be the number of (x, Y)-bridges other than X containing an edge of S. From the definition of $\sigma(P, e)$, we deduce that $\sigma(P, e) \geq$

$\sigma_X(P_x, e) + k$ for any $e \in E(P_x)$. So

$$\begin{aligned}\sum_{e\in E(P)} 2^{\sigma(P,e)}\omega(e) &\geq \sum_{e\in E(P_y)} 2^{\sigma_Y(P_y,e)}\omega(e) + \sum_{e\in E(P_x)} 2^{\sigma(P,e)}\omega(e)\\ &\geq \omega(Y) + \sum_{e\in E(P_x)} 2^{k+\sigma_X(P_x,e)}\omega(e)\\ &\geq \omega(Y) + 2^k\omega(X)\\ &\geq \omega(H),\end{aligned}$$

where the last inequality holds since $2^k\omega(X) \geq (k+1)\omega(X) \geq \omega(H-Y)$. ∎

For each ordered edge pair (e, f) of H and $S \subseteq E(H)$, let $r(e, f; H)$ denote the maximum number of S-rungs of a ladder with top e and bottom f. Clearly, $r(e, f; H) \geq \sigma(P, e)$ for any x-y path P passing through e, where $f = xy$.

Corollary 4.2. *Let H be a 2-connected graph, let $f = xy \in E(H)$, let $\omega : E(H) \mapsto \mathbb{R}^+$, and let $S = \{e \in E(H) : \omega(e) > 0\}$. Suppose $r(e, f; H) = 0$ for some $e \in E(H)$. Then there exists an x-y path P passing through e in H such that*

$$\sum_{g\in E(P)} 2^{\sigma(P,g)}\omega(g) \geq \omega(H).$$

Proof. Let P be the x-y path as exhibited in Theorem 4.1. If $e \in E(P)$, then we are done. So we assume $e \notin E(P)$. Since H is 2-connected, it contains two vertex-disjoint paths Q_1, Q_2 from the ends of e to P. Let v_1 and v_2 be the ends of Q_1 and Q_2 on P, respectively, and let R be the path obtained from $P \cup Q_1 \cup Q_2$ by deleting all vertices on $P(v_1, v_2)$. Since $r(e, f; H) = 0$, we have $P[v_1, v_2] \cap S = \emptyset$; that is, $w(g) = 0$ for all edges g on $P[v_1, v_2]$. So $\sum_{g\in E(R)} 2^{\sigma(R,g)}\,\omega(g) \geq \sum_{g\in E(P)} 2^{\sigma(P,g)}\,\omega(g) \geq \omega(H)$. ∎

5 Proof of Theorem 2.5(a)

The following lemma serves as the induction step in the proof of Theorem 2.5(a). Recall that $b = 1729$ and $\beta = \log_b 2$.

Lemma 5.1. *Suppose $n > b^{t(t-1)}$, $t \geq 2$, and Theorem 2.5 holds for graphs with at most $n-1$ vertices and for graphs containing no $K_{3,t}$-minors. Then Theorem 2.5(a) holds for graphs with n vertices.*

Proof. Let G be a 3-connected n-vertex graph with $\tau(G) \leq t$, let x, y, z be three distinct vertices of G with $xz,\ yz \in E(G)$, and let $H = (G - z) + xy$ (so $|H| = n - 1$). Suppose Algorithm 2.2 has been applied to $(H; xy)$. Our objective is to prove that there exists an x-y path in $G - z$ of length at least $\alpha(t)\left(\delta(t, H)(n-1)\right)^{\beta}$, where $\delta(t, H)$ is as defined in (2.2) with $e_0 = xy$ in place of f.

In our proof we shall frequently use the following identities:

$$\alpha(t-1) = \alpha(t)4^{t-1} \quad \text{and} \quad 4^{(t-1)/\beta} = 2^{2(t-1)\log_2 b} = b^{2(t-1)}. \tag{5.1}$$

Claim 5.1. *We may assume $\tau(G) = t$.*

Otherwise, $\tau(G) \leq t-1$, so G contains no $K_{3,t}$-minors. Hence the induction hypothesis of Theorem 2.5(a) guarantees the existence of an x-y path P in $G - z$ such that

$$\begin{aligned}\ell(P) &\geq \alpha(t-1)\left(\delta(t-1,H)|H|\right)^{\beta} \\ &= \alpha(t)\left(b^{2(t-1)}\delta(t-1,H)|H|\right)^{\beta} \quad \text{(by (5.1))} \\ &\geq \alpha(t)\left(\delta(t,H)|H|\right)^{\beta} \quad \text{(by Lemma 2.4(iii) and since } b = 1729\text{)}.\end{aligned}$$

Claim 5.2. *We may assume that H is not 3-connected.*

Suppose on the contrary that H is 3-connected. Then $\theta(H) = \phi(H) = 0$ (see the comment above (2.2)), so $\delta(t,H) = 1$. Hence, by the induction hypothesis of Theorem 2.5(c), there exists an x-y path P in H (hence in $G - z$) with $\ell(P) \geq \alpha(t)|H|^{\beta} = \alpha(t)\left(\delta(t,H)|H|\right)^{\beta}$.

For each $f = uv \in \Psi(H)$, let H_f be as defined in Algorithm 2.2 and let $G_f = G[V(H_f) \cup z] + \{uz, vz\}$. Recall that in Algorithm 2.2 we set $e_0 = xy$.

Claim 5.3. *If $f \neq e_0$, then H_f contains a u-v path of length at least $\alpha(t)\left(\delta(t,H)|H_f|\right)^{\beta}$.*

Since $f \neq e_0$, we have $|G_f| < |G|$. By Lemma 2.3(iv), G_f is a 3-connected minor of G. Let s be any integer such that $\tau(G_f) \leq s$. Then the induction hypothesis of Theorem 2.5(a) implies the existence of a u-v path P in H_f with $\ell(P) \geq \alpha(s)\left(\delta(s,H_f)|H_f|\right)^{\beta}$.

Suppose $\tau(G_f) = \tau(G)$. Set $s = t$. By Lemma 2.4(iv), we get $\delta(t,H_f) \geq \delta(t,H)$. Thus $\ell(P) \geq \alpha(t)\left(\delta(t,H)|H_f|\right)^{\beta}$, as desired. So we may assume that $\tau(G_f) < \tau(G) = t$. Set $s = t-1$. Then the same argument used in the proof of Claim 5.1 implies

$$\ell(P) \geq \alpha(t-1)\left(\delta(t-1,H_f)|H_f|\right)^{\beta} \geq \alpha(t)\left(\delta(t,H)|H_f|\right)^{\beta}.$$

Suppose Case 2 or Case 3 of Algorithm 2.2 occurs; see the descriptions. Set $\hat{H}_i = H_i + a_i b_i$, where $a_i b_i = x_i x_{i+1}$ in Case 2 and $a_i b_i = u_i v_i$ in Case 3, and set $G_i = G[\hat{H}_i \cup \{z\}] + a_i z + b_i z$. By the hypotheses of Cases 2 and Case 3, $\{x,y\}$ is not a cutset of H, so $\{a_i, b_i\} \neq \{x,y\}$. Using exactly the same proof as that of Lemma 2.3(iv), we see that G_i is a 3-connected minor of G. In particular, $\tau(G_i) \leq \tau(G)$. By applying Algorithm 2.2 directly to the input $(\hat{H}_i; a_i b_i)$ (that is, with $(\hat{H}_i; a_i b_i)$ in place of $(H; e_0)$), we can define $\theta(\hat{H}_i)$, $\phi(\hat{H}_i)$, and $\delta(t,\hat{H}_i)$ accordingly. Now the same argument of Lemma 2.4(iv) implies that if $\tau(G_i) = \tau(G)$, then $\delta(t,H) \leq \delta(t,\hat{H}_i)$ for any $t \geq \tau(G)$. Finally, imitating the proof of Claim 5.3, we get the following statement.

Claim 5.4. *There exists an $a_i - b_i$ path in H_i of length at least $\alpha(t)\left(\delta(t,H)|H_i|\right)^{\beta}$.*

Claim 5.5. *For any $f = uv \in \Psi(H)$ with $\tau(G_f) \leq t-1$, we may assume $|H_f| < \frac{|H|}{8t^2}$.*

Suppose $|H_f| \geq \frac{|H|}{8t^2}$. By Lemma 2.3(iv), G_f is a 3-connected minor of G. In view of Claim 5.1, $G_f \neq G$, so $|G_f| < |G|$. Thus the induction hypothesis of Theorem 2.5(a) yields a u-v path P in H_f such that

$$\begin{aligned}
\ell(P) &\geq \alpha(t-1)\left(\delta(t-1, H_f)|H_f|\right)^{\beta} \\
&\geq \alpha(t-1)\left(\frac{|H_f|}{27}\right)^{\beta} \quad \text{(by Lemma 2.4(iii))} \\
&\geq \alpha(t)\left(\frac{b^{2(t-1)}}{216t^2}|H|\right)^{\beta} \quad \text{(by (5.1) and since } |H_f| \geq \tfrac{|H|}{8t^2}) \\
&> \alpha(t)\left(\delta(t, H)|H|\right)^{\beta} \quad \text{(since } \tfrac{b^{2(t-1)}}{216t^2} > 1 \geq \delta(t,H) \text{ for } t \geq 2).
\end{aligned}$$

Since H is 2-connected, it contains two vertex-disjoint paths Q_1 and Q_2 from $\{x, y\}$ to $\{u, v\}$. So $Q_1 \cup P \cup Q_2$ is an x-y path in $G - z$ of length at least $\alpha(t)\left(\delta(t, H)|H|\right)^{\beta}$.

Claim 5.6. *We may assume that $\{x, y\}$ is not a cutset of H; so Case 1 of Algorithm 2.2 cannot occur.*

Suppose the contrary: $\{x, y\}$ is a cutset of H. As described in Case 1 of Algorithm 2.2, let $B_1, B_2, \ldots, B_m$ be all the nontrivial (x, y)-bridges, let $H_{e_i} = B_i + e_i$ for each i, where e_i is a virtual edge between x and y, and let $G_{e_i} = G[H_{e_i} \cup z] + \{xz, yz\}$. Using these B_i's, it is easy to see that G contains a $K_{3,m}$-minor rooted at $\{x, y, z\}$, so $m \leq \tau(G) = t$ by Claim 5.1. Renaming subscripts if necessary, we assume that $|H_{e_1}| = \max\{|H_{e_i}| : 1 \leq i \leq m\}$. Then $|H_{e_1}| \geq (|H| - 2)/m + 2 \geq |H|/t$. From Claim 5.5 it follows that $\tau(G_{e_1}) = t$. Set $k := |\{i \geq 2 \ : \ \tau(G_{e_i}) = t\}|$. Without loss of generality, we may assume that $\tau(G_{e_i}) = t$ for $i = 2, 3, \ldots, k + 1$. Clearly, $m \geq k + 1$ and $\theta(H) \geq \theta(H_{e_1}) + k$. We claim that

$$\delta(t, H) \leq \frac{1}{3^{\theta(H_{e_1})+k}}\left(1 - \frac{\phi(H_{e_1}) - \theta(H_{e_1}) + m - 1 - k}{3t}\right). \tag{5.2}$$

If $\theta(H) = \theta(H_{e_1}) + k$ then, by (2.1), we have $\phi(H) \geq \phi(H_{e_1}) + (m - 1)$. Thus (5.2) follows from the definition of $\delta(t, H)$. So we may assume $\theta(H) \geq \theta(H_{e_1}) + k + 1$. Then

$$\delta(t, H) \leq \frac{1}{3^{\theta(H)}} \leq \frac{1}{3^{\theta(H_{e_1})+k+1}} \leq \frac{1}{3^{\theta(H_{e_1})+k}}\left(1 - \frac{\phi(H_{e_1}) + m}{3t}\right) \leq \text{the RHS of (5.2)},$$

where the third inequality holds since $m \leq t$ and $\phi(H_{e_1}) \leq \tau(G_{e_1}) \leq t$ by Lemma 2.4(i).

Observe that the RHS of (5.2) is at most

$$\frac{1}{3^{\theta(H_{e_1})+k}}\left(1 - \frac{\phi(H_{e_1}) - \theta(H_{e_1})}{3t}\right)\left(1 - \frac{m - 1 - k}{3t}\right) = \delta(t, H_{e_1})\left(\frac{1}{3^k}\left(1 - \frac{m - 1 - k}{3t}\right)\right).$$

Hence

$$\delta(t, H) \leq \delta(t, H_{e_1})\left(\frac{1}{3^k}\left(1 - \frac{m - 1 - k}{3t}\right)\right). \tag{5.3}$$

By Claim 5.5, $|H_{e_j}| \leq \frac{|H|}{8t^2}$ for any j with $k + 2 \leq j \leq m$. It follows from the maximality of $|H_{e_1}|$ that

$$|H_{e_1}| \geq \frac{1}{k+1}\left(1 - \frac{m - 1 - k}{8t^2}\right)|H| > \frac{1}{3^k}\left(1 - \frac{m - 1 - k}{3t}\right)|H|. \tag{5.4}$$

By the induction hypothesis of Theorem 2.5(a), there exists an x-y path P in H_{e_1} such that $\ell(P) \geq \alpha(t)\left(\delta(t, H_{e_1})|H_{e_1}|\right)^{\beta}$. From (5.3) and (5.4), we conclude that

$$\ell(P) \geq \alpha(t)\left(\delta(t, H_{e_1})\frac{1}{3^k}\left(1 - \frac{m-1-k}{3t}\right)|H|\right)^{\beta} \geq \alpha(t)\left(\delta(t, H)|H|\right)^{\beta}.$$

Recall that H^* is the leading block H^* output by Algorithm 2.2.

Claim 5.7. *We may assume that H^* is not a multicycle; so Case 2 of Algorithm 2.2 cannot occur and H^* is 3-connected by Lemma 2.3(iii).*

Suppose to the contrary that H^* is a multicycle. By Claim 5.6, $\{x, y\}$ is not a cutset of H. So xy is the unique edge in H^* with ends x and y. Therefore, $H^* - xy$ can be obtained from a simple path $a_0a_1 \ldots a_k$ by adding parallel edges, where $a_0 = x$ and $a_k = y$. For each i with $0 \leq i \leq k-1$, let $\hat{H}_i$ be the graph as defined right above Claim 5.4, where $b_i = a_{i+1}$. By Claim 5.4, there exists an a_i-a_{i+1} path P_i in H_i such that $\ell(P_i) \geq \alpha(t)\left(\delta(t, H)|H_i|\right)^{\beta}$. Concatenating all these P_i, we obtain an $x - y$ path P with

$$\ell(P) \geq \sum_{i=0}^{k-1}\alpha(t)\left(\delta(t, H)|H_i|\right)^{\beta} \geq \alpha(t)\left(\delta(t, H)\sum_{i=0}^{k-1}|H_i|\right)^{\beta} \geq \alpha(t)\left(\delta(t, H)|H|\right)^{\beta},$$

where the second inequality follows from Corollary 3.6.

For $i = 1, 2$, set $\Psi_i := \Psi_i(H) \cap E(H^*) - \{e_0\}$, where $e_0 = xy$, and define a weight function ω_i: $E(H^*) \mapsto \mathbb{R}^+$ as follows:

$$\omega_i(f) = \begin{cases} |H_f|, & \text{if } f \in \Psi_i \\ 0, & \text{otherwise.} \end{cases}$$

In addition, set $\omega_i(H^*) := \sum_{f \in \Psi_i} |H_f|$.

Claim 5.8. *We may assume that $\omega_2(H^*) < \frac{|H|}{9t}$.*

Otherwise, $\omega_2(H^*) \geq \frac{|H|}{9t}$. By Theorem 4.1 (with respect to ω_2), there exists an x-y path Q in H^* such that $\sum_{e \in E(Q)} 2^{\sigma(Q,e)}\omega_2(e) \geq \omega_2(H^*)$, where $\sigma(Q, e)$ is the maximum number of Ψ_2-rungs of a ladder in H^* generated by Q with top e and bottom xy. Since $\tau(G) = t$, we have $\sigma(Q, e) \leq t$, which implies $\sum_{e \in E(Q)} \omega_2(e) \geq \omega_2(H^*)/2^t$. So $\sum_{e \in E(Q) \cap \Psi_2} |H_e| \geq \omega_2(H^*)/2^t \geq |H|/(9t2^t)$.

For each $e \in E(Q) \cap \Psi_2$, there holds $\tau(G_e) \leq t - 1$. So the induction hypothesis of Theorem 2.5(a) guarantees the existence of a path P_e in H_e between the ends of e such that

$$\ell(P_e) \geq \alpha(t-1)\left(\delta(t-1, H_e)|H_e|\right)^{\beta} \geq \alpha(t-1)\left(\frac{|H_e|}{27}\right)^{\beta} = \alpha(t)\left(\frac{b^{2(t-1)}}{27}|H_e|\right)^{\beta},$$

where the second inequality follows from Lemma 2.4(iii) and the equality from (5.1).

Concatenating these P_e and all edges in $E(Q) - \Psi_2$, we obtain an x-y path that leads to an x-y path P in H with

$$\begin{aligned}
\ell(P) &\geq \sum_{e\in E(Q)\cap\Psi_2} \alpha(t)\left(\frac{b^{2(t-1)}}{27}|H_e|\right)^\beta \\
&\geq \alpha(t)\left(\frac{b^{2(t-1)}}{27}\sum_{e\in E(Q)\cap\Psi_2}|H_e|\right)^\beta \quad \text{(by Corollary 3.6)} \\
&\geq \alpha(t)\left(\frac{b^{2(t-1)}}{27\cdot 9t\cdot 2^t}|H|\right)^\beta \\
&\geq \alpha(t)|H|^\beta \quad (\text{since } b^{2(t-1)} \geq 27\cdot 9t\cdot 2^t) \\
&\geq \alpha(t)\left(\delta(t,H)|H|\right)^\beta \quad (\text{since } \delta(t,H)\leq 1).
\end{aligned}$$

Claim 5.9. *We may assume $\theta(H) \geq 1$; so $\delta(t,H) \leq 1/3$ by Lemma 2.4(iii).*

Suppose $\theta(H) = 0$. By Claim 5.2 and Claim 5.6, $\Psi(H) - \{f\} \neq \emptyset$; so $\phi(H) \geq 1$ (by definition). Thus

$$\delta(t,H) = \frac{1}{3^{\theta(H)}}\left(1 - \frac{\phi(H)-\theta(H)}{3t}\right) = 1 - \frac{\phi(H)}{3t} \leq \frac{3t-1}{3t}.$$

Since $\Psi_1(H) = \emptyset$ (as $\theta(H) = 0$), by Claim 5.8 we have

$$|H^*| \geq |H| - \omega_2(H^*) \geq |H| - \frac{|H|}{9t} \geq \frac{3t-1}{3t}(n-1) \geq \delta(t,H)(n-1).$$

As H^* is a 3-connected minor of G (by Lemma 2.3(ii) and Claim 5.7), the induction hypothesis of Theorem 2.5(c) gives an x-y path P in H^*, which clearly leads to an x-y path Q in H with $\ell(Q) \geq \ell(P) \geq \alpha(t)|H^*|^\beta \geq \alpha(t)\left(\delta(t,H)(n-1)\right)^\beta$.

Claim 5.10. *We may assume that $|H^*| < \frac{|H|}{3}$.*

Assume $|H^*| \geq |H|/3$. Since H^* is a 3-connected minor of G (by Lemma 2.3(ii) and Claim 5.7), the induction hypothesis of Theorem 2.5(c) gives an x-y path P in H^*, which clearly leads to an x-y path Q in H, such that $\ell(Q) \geq \ell(P) \geq \alpha(t)|H^*|^\beta \geq \alpha(t)\left(\frac{1}{3}|H|\right)^\beta \geq \alpha(t)\left(\delta(t,H)|H|\right)^\beta$ (by Claim 5.9).

Since $t \geq 2$, combining Claims 5.10 and 5.8 we obtain

$$\omega_1(H^*) = \sum_{e\in\Psi_1}|H_e| \geq |H| - |H^*| - \omega_2(H^*) \geq (1 - \frac{1}{3} - \frac{1}{9t})|H| > |H|/2. \tag{5.5}$$

By Theorem 4.1, there exists an x-y path Q in H^* such that

$$\sum_{e\in E(Q)} 2^{\sigma(Q,e)}\omega_1(e) \geq \omega_1(H^*), \tag{5.6}$$

where $\sigma(Q,e)$ is the maximum number of Ψ_1-rungs of a ladder in H^* generated by Q with top e and bottom $e_0 = xy$. We shall use Q to produce a desired path in H, by comparing $|H^*|$, $w_1(H^*)$ and $w_2(H^*)$.

Claim 5.11. *For each $e \in E(Q) \cap \Psi_1$, there holds $\delta(t, H_e) \geq \delta(t, H) \cdot 3^{\sigma(Q,e)}$.*

Since $e \in \Psi_1(H)$, we have $\tau(G_e) = \tau(G)$. It is then a routine matter to check that $\theta(H) \geq \theta(H_e) + \sigma(Q, e)$. To justify the claim, we distinguish between two cases. If $\theta(H) \geq \theta(H_e) + \sigma(Q, e) + 1$, then

$$\delta(t, H) \leq \frac{1}{3^{\theta(H_e)+\sigma(Q,e)+1}} \leq \frac{1}{3^{\sigma(Q,e)}3^{\theta(H_e)}} \left(1 - \frac{\phi(H_e) - \theta(H_e)}{3t}\right) = \delta(t, H_e)/3^{\sigma(Q,e)},$$

where the last inequality holds since $\phi(H_e) \leq t$ by Lemma 2.4(i). So we assume $\theta(H) = \theta(H_e) + \sigma(Q, e)$. Now from (2.1) we deduce that $\phi(H) \geq \phi(H_e) + \sigma(Q, e)$. Thus $\phi(H) - \theta(H) \geq (\phi(H_e) + \sigma(Q, e)) - (\theta(H_e) + \sigma(Q, e)) \geq \phi(H_e) - \theta(H_e)$. Hence the desired inequality follows instantly as in the previous case, completing the proof of Claim 5.11.

Let g be an edge on Q such that $3^{\sigma(Q,g)}\omega_1(g) = \max_{e \in E(Q)} \{3^{\sigma(Q,e)}\omega_1(e)\}$, and set

$$\lambda := \sum_{e \in E(Q)-\{g\}} 3^{\sigma(Q,e)}\omega_1(e).$$

For each $e = uv \in E(Q) \cap \Psi_1$, let $G_e = G[V(H_e) \cup \{z\}] + \{zu, zv, uv\}$. By Lemma 2.3(iv), G_e is a 3-connected minor of G. So by the induction hypothesis of Theorem 2.5(a), there exists a path P_e in H_e between the ends of e such that

$$\ell(P_e) \geq \alpha(t) \left(\delta(t, H_e)|H_e|\right)^{\beta} \geq \alpha(t) \left(\delta(t, H)3^{\sigma(Q,e)}\omega_1(e)\right)^{\beta}, \tag{5.7}$$

where the second inequality follows from Claim 5.11.

Claim 5.12. *We may assume that $\lambda < \frac{1}{b-2}(|H^*| + \omega_2(H^*))$.*

Otherwise, $(b - 2)\lambda \geq |H^*| + \omega_2(H^*)$; so $w_1(H^*) + (b - 2)\lambda \geq |H^* + w_2(H^*)$. Concatenating all P_e, with $e \in E(Q) \cap \Psi_1$, and paths in H_e corresponding to all edges $e \in E(Q) - \Psi_1$, we obtain an x-y path P in $G - z$ such that

$$\begin{aligned}
\ell(P) &\geq \sum_{e \in E(Q)\cap\Psi_1} \ell(P_e) \\
&\geq \alpha(t) \sum_{e \in E(Q)\cap\Psi_1} \left(\delta(t, H)3^{\sigma(Q,e)}\omega_1(e)\right)^{\beta} \quad \text{(by (5.7))} \\
&\geq \alpha(t) \left(\delta(t, H) \left(3^{\sigma(Q,g)}\omega_1(g) + (b - 1) \sum_{e \in E(Q)-\{g\}} 3^{\sigma(Q,e)}\omega_1(e)\right)\right)^{\beta} \quad \text{(by Corollary 3.6)} \\
&= \alpha(t) \left(\delta(t, H) \left(3^{\sigma(Q,g)}\omega_1(g) + (b - 1)\lambda\right)\right)^{\beta} \\
&\geq \alpha(t) \left(\delta(t, H) \left(\omega_1(H^*) + (b - 2)\lambda\right)\right)^{\beta} \quad \text{(by (5.6))} \\
&\geq \alpha(t) \left(\delta(t, H)|H|\right)^{\beta}.
\end{aligned}$$

Claim 5.13. *We may assume* $|H^*| < \omega_2(H^*)$.

Suppose $|H^*| \geq \omega_2(H^*)$. Then by Claim 5.12 and Claim 5.10, we have $\lambda \leq \frac{2|H^*|}{b-2} \leq \frac{2|H|}{3(b-2)}$. From the definition of λ, (5.6) and (5.5), it follows that $3^{\sigma(Q,g)}\omega_1(g) \geq \omega_1(H^*) - \lambda \geq \frac{|H|}{2} - \frac{2|H|}{3(b-2)} \geq \frac{|H|}{3}$. So by Claim 5.10, we have

$$3^{\sigma(Q,g)}\omega_1(g) \geq |H^*|. \tag{5.8}$$

Let P_g be the path as specified in (5.7) with $e = g$. By Lemma 2.3(ii) and Claim 5.7, H^* is a 3-connected minor of G. So the induction hypothesis of Theorem 2.5(b) gives an x-y path Q_g passing g in H^* such that $\ell(Q_g) \geq \alpha(t)(|H^*|/28)^\beta + 1$. Let P be the x-y path obtained from Q_g by replacing g with P_g. Then

$$\begin{aligned}
\ell(P) &= \ell(P_g) + \ell(Q_g) - 1 \\
&\geq \alpha(t)\left(\delta(t,H)3^{\sigma(Q,g)}\omega_1(g)\right)^\beta + \alpha(t)\left(|H^*|/28\right)^\beta \\
&\geq \alpha(t)\left(\delta(t,H)3^{\sigma(Q,g)}\omega_1(g)\right)^\beta + \alpha(t)\left(\delta(t,H)|H^*|/28\right)^\beta \quad \text{(because } \delta(t,H) \leq 1\text{)} \\
&\geq \alpha(t)\left(\delta(t,H)\left(3^{\sigma(Q,g)}\omega_1(g) + (b-1)(|H^*|/28)\right)\right)^\beta \quad \text{(by (5.8) and Corollary 3.6)} \\
&\geq \alpha(t)\left(\delta(t,H)\left(3^{\sigma(Q,g)}\omega_1(g) + \lambda + |H^*| + \omega_2(H^*)\right)\right)^\beta \quad \text{(by Claim 5.12 ans since } b = 1729\text{)} \\
&\geq \alpha(t)\left(\delta(t,H)\left(\omega_1(H^*) + |H^*| + \omega_2(H^*)\right)\right)^\beta \quad \text{(by (5.6))} \\
&\geq \alpha(t)\left(\delta(t,H)|H|\right)^\beta.
\end{aligned}$$

Path P obviously leads to an x-y path R in H with $\ell(R) \geq \ell(P) \geq \alpha(t)\left(\delta(t,H)|H|\right)^\beta$.

Claim 5.14. *We may assume that* $\sigma(Q,g) = 0$, $\omega_1(g) \geq \left(1 - \frac{1}{3t}\right)|H|$, *and* $\delta(t,H_g) < \delta(t,H)/\left(1 - \frac{1}{3t}\right)$.

By (5.6), $\sum_{e \in E(Q)} 2^{\sigma(Q,e)}\omega_1(e) \geq \omega_1(H^*) \geq |H| - |H^*| - \omega_2(H^*)$. So by Claims 5.12 and 5.13,

$$2^{\sigma(Q,g)}\omega_1(g) \geq |H| - |H^*| - \omega_2(H^*) - \lambda \geq |H| - \frac{b-1}{b-2}(|H^*| + \omega_2(H^*)) \geq |H| - \frac{2(b-1)}{b-2}\omega_2(H^*).$$

Since $\omega_2(H^*) < \frac{|H|}{9t}$ (by Claim 5.8), $2^{\sigma(Q,g)}\omega_1(g) \geq \left(1 - \frac{2(b-1)}{9t(b-2)}\right)|H|$. Hence

$$2^{\sigma(Q,g)}\omega_1(g) \geq \left(1 - \frac{1}{3t}\right)|H|. \tag{5.9}$$

Let P_g be the path as exhibited in (5.7) with $e = g$. Then by (5.7) and (5.9),

$$\ell(P_g) \geq \alpha(t)\left(\delta(t,H)3^{\sigma(Q,g)}\omega_1(g)\right)^\beta \geq \alpha(t)\left(\delta(t,H)(\frac{3}{2})^{\sigma(Q,g)}(1 - \frac{1}{3t})|H|\right)^\beta.$$

Suppose $\sigma(Q,g) \geq 1$. Then $\ell(P_g) \geq \alpha(t)\left(\delta(t,H)|H|\right)^\beta$. Let R_1 and R_2 be two vertex-disjoint paths in H from $\{x,y\}$ to the two ends of g (and internally disjoint from H_g). Clearly $R_1 \cup P_g \cup R_2$ leads to an x-y path in H with length at least $\alpha(t)\,(\delta(t,H)|H|)^{\beta}$. So we may assume $\sigma(Q,g)=0$. Then by (5.9), $|H_g|=\omega_1(g)\ge(1-\frac{1}{3t})|H|$. If $\delta(t,H_g)\ge\delta(t,H)/\left(1-\frac{1}{3t}\right)$, then

$$\ell(P_g)\ge\alpha(t)\,(\delta(t,H_g)|H_g|)^{\beta}\ge\alpha(t)\left(\frac{\delta(t,H)}{1-\frac{1}{3t}}\cdot\left(1-\frac{1}{3t}\right)|H|\right)^{\beta}=\alpha(t)\,(\delta(t,H)|H|)^{\beta}.$$

Clearly P_g leads to a desired path for the lemma. So we may assume $\delta(t,H_g)<\delta(t,H)/\left(1-\frac{1}{3t}\right)$.

Claim 5.15. *$\theta(H_g)=\theta(H)$ and $\phi(H_g)=\phi(H)$.*

By Claim 5.14, we have $\omega_1(g)\neq 0$. So $g\in\Psi_1(H)$ and hence $\tau(G_g)=\tau(G)$. It follows that $\theta(H_g)\le\theta(H)$. If $\theta(H)\ge\theta(H_g)+1$ then, by Lemma 2.4(i),

$$\delta(t,H_g)=\frac{1}{3^{\theta(H_g)}}\left(1-\frac{\phi(H_g)-\theta(H_g)}{3t}\right)\ge\frac{1}{3^{\theta(H_g)}}\left(1-\frac{t}{3t}\right)\ge\frac{2}{3^{\theta(H)}}\ge 2\delta(t,H)>\frac{\delta(t,H)}{1-\frac{1}{3t}},$$

contradicting Claim 5.14. So $\theta(H_g)=\theta(H)$.

From $\theta(H_g)=\theta(H)$ and (2.1), we deduce that $\phi(H_g)\le\phi(H)$. If $\phi(H)\ge\phi(H_g)+1$, then $\phi(H)-\theta(H)\ge\phi(H_g)-\theta(H_g)+1$. It follows that

$$\delta(t,H)\le\frac{1}{3^{\theta(H_g)}}(1-\frac{\phi(H_g)-\theta(H_g)+1}{3t})\le\frac{1}{3^{\theta(H_g)}}(1-\frac{\phi(H_g)-\theta(H_g)}{3t})(1-\frac{1}{3t})=\delta(t,H_g)(1-\frac{1}{3t}).$$

This contradicts Claim 5.14, and so $\phi(H_g)=\phi(H)$, proving Claim 5.15.

Finally, we define the third weight function ω_3: $E(H^*)\mapsto\mathbb{R}^+$ as follows:

$$\omega_3(f)=\begin{cases}|H_f|, & \text{if } f\in\Psi_2\cup\{g\}\\ 0, & \text{otherwise.}\end{cases}$$

Notice that ω_2 and ω_3 are identical except that $\omega_2(g)=0$ while $\omega_3(g)=|H_g|$. From Claim 5.15 ($\phi(H_g)=\phi(H)$), we deduce that no Ψ_2-rungs exist for any ladder in H^* with top g and bottom $e_0=xy$. So, using the notation introduced right above Corollary 4.2, we obtain $r(g,e_0;H^*)=0$. By this corollary, there exists an x-y path R passing through g in H^* such that $\sum_{e\in E(R)}2^{\sigma(R,e)}\omega_3(e)\ge\omega_3(H^*)=\omega_3(g)+\omega_2(H^*)$, where $\sigma(R,e)$ is the maximum number of Ψ_2-rungs of a ladder in H^* generated by R with top e and bottom $e_0=xy$. Since $r(g,e_0;H^*)=0$, we have $\sigma(R,g)=0$. So $\sum_{e\in E(R)}2^{\sigma(R,e)}\omega_2(e)\ge\omega_2(H^*)$, because $\omega_2(g)=0$. As $\tau(G)\le t$, it is easy to see that $\sigma(R,e)\le t-1$ for each $e\in E(R)\cap\Psi_2$ (recall that $\sigma(R,e)$ does not count e). Hence

$$\sum_{e\in E(R)}\omega_2(e)\ge\omega_2(H^*)/2^{t-1}. \tag{5.10}$$

For each $e\in E(R)\cap\Psi_2$, we have $\tau(G_e)\le\tau(G)-1\le t-1$. So the induction hypothesis of Theorem 2.5(a) gives a path R_e in H_e between the ends of e such that

$$\ell(R_e)\ge\alpha(t-1)\,(\delta(t-1,H_e)|H_e|)^{\beta}\ge\alpha(t-1)\left(\frac{|H_e|}{27}\right)^{\beta}=\alpha(t)4^{t-1}\left(\frac{\omega_2(e)}{27}\right)^{\beta},$$

where the second inequality follows from Lemma 2.4(iii) and the equality follows from (5.1). Let P_g be the path as specified in (5.7) with $e = g$. Concatenating P_g, all these R_e, and paths in H_e corresponding to all edges $e \in E(R) - (\Psi_2 \cup \{g\})$, we obtain an x-y path T in H such that

$$\ell(T) \geq \alpha(t) \left((\delta(t,H)\omega_1(g))^\beta + \sum_{e \in E(R)} 4^{t-1} \left(\frac{\omega_2(e)}{27} \right)^\beta \right). \tag{5.11}$$

By Lemma 2.4(iii) and Claim 5.14, $\delta(t,H)\omega_1(g) \geq \frac{\omega_1(g)}{27} \geq \frac{1}{27}(1 - \frac{1}{3t})|H| = \frac{3t-1}{81t}|H|$. By Claim 5.8, $\omega_2(H^*) < \frac{|H|}{9t}$. So

$$\delta(t,H)\omega_1(g) \geq \frac{3t-1}{9}\omega_2(H^*) \geq \frac{1}{2}\omega_2(H^*) \geq \frac{\omega_2(e)}{27} \tag{5.12}$$

for any $e \in E(R)$. Let us view $4^{t-1}\left(\frac{\omega_2(e)}{27}\right)^\beta$ as the sum of 4^{t-1} terms, each being $\left(\frac{\omega_2(e)}{27}\right)^\beta$. Applying Corollary 3.6 to the RHS of (5.11) and using (5.12), we have

$$\ell(T) \geq \alpha(t) \left(\delta(t,H)\omega_1(g) + 4^{t-1}(b-1) \sum_{e \in E(R)} \frac{\omega_2(e)}{27} \right)^\beta.$$

Plugging in (5.10) and using the inequality $(b-1)4^{t-1}/(27 \cdot 2^{t-1}) \geq 3$, we obtain

$$\begin{aligned}
\ell(T) &\geq \alpha(t) \left(\delta(t,H)\omega_1(g) + 4^{t-1}(b-1) \sum_{e \in E(R)} \frac{\omega_2(e)}{27} \right)^\beta \\
&\geq \alpha(t) \left(\delta(t,H)\omega_1(g) + 4^{t-1}(b-1)w_2(H^*)/2^{t-1} \right)^\beta \quad \text{(by (5.10))} \\
&> \alpha(t) \left(\delta(t,H) \left(\omega_1(g) + 3\omega_2(H^*) \right) \right)^\beta \\
&\geq \alpha(t) \left(\delta(t,H) \left(|H_g| + |H^*| + \omega_2(H^*) + \lambda \right) \right)^\beta \quad \text{(by Claims 5.12 and 5.13)} \\
&\geq \alpha(t) \left(\delta(t,H) \left(|H^*| + \omega_1(H^*) + \omega_2(H^*) \right) \right)^\beta \\
&\geq \alpha(t) \left(\delta(t,H)|H| \right)^\beta,
\end{aligned}$$

where the second last inequality holds because $\sigma(Q,g) = 0$ (by Claim 5.14). From the definition of λ and (5.6) it follows that $\lambda + |H_g| \geq \omega_1(H^*)$. This completes the proof of Lemma 5.1. ■

6 Proof of Theorem 2.5(b)

Let us establish the following lemma, which serves as the induction step for proving Theorem 2.5(b).

Lemma 6.1. *Suppose $n > b^{t(t-1)}$, $t \geq 2$, and Theorem 2.5 holds for graphs with at most $n-1$ vertices and for graphs containing no $K_{3,t}$-minors. Then Theorem 2.5(b) holds for graphs with n vertices.*

Proof. We may assume that e and f are nonadjacent, for otherwise, symmetry allows us to assume that y is the common end of both e and f. Let w be the other end of e and let $H := G - y$. Then, by

Lemma 5.1, H contains an x-w path P with length $\ell(P) \geq \alpha(t)(\delta(t,H)|H|)^{\beta} \geq \alpha(t)(\frac{n-1}{27})^{\beta} \geq \alpha(t)(\frac{n}{28})^{\beta}$ for $n \geq 28$, where the second inequality follows from Lemma 2.4(iii). Let Q be the path obtained from P by appending the edge e. Clearly, Q is an x-y path through e with length at least $\alpha(t)(n/28)^{\beta}+1$ in G.

As G is 3-connected, it contains an x-y path Q through e. Let Q_x and Q_y be the components of $Q-e$ containing x and y, respectively. Let $x_0X_0x_1X_1x_2\dots x_pX_px_{p+1}$ denote the chain of blocks in $G-V(Q_y)$ from x to x_{p+1}, where $x_0 = x$ and x_{p+1} is incident with e. Let $X = \bigcup_{i=1}^{p} X_i$. Clearly, $Q_x \subseteq X$.

Since G is 3-connected, $U_i := G - V(X_i)$ is connected for each i with $0 \leq i \leq p$. From Lemma 3.3, we deduce that $X_i^* := G/U_i$ is either a triangle or a 3-connected minor of G. Let u_i denote the vertex of X_i^* resulted from the contraction of U_i. Clearly, $u_ix_i, u_ix_{i+1} \in E(X_i^*)$. Since $|U_i| \geq 2$, we have $|X_i^*| < n$.

From 3-connectedness of G, we see that $Y := G - V(X)$ is a chain of blocks in $G - V(X)$. Suppose Y is $y_0Y_0y_1Y_1y_2\dots y_qY_qy_{q+1}$, where y_{q+1} is incident with e and $y_0 = y$. Since G is 3-connected, $W_j := G - V(Y_j)$ is connected for each j with $0 \leq j \leq q$. By Lemma 3.3, $Y_j^* := G/W_j$ is either a triangle or a 3-connected minor of G. Let w_j denote the vertex of Y_j^* resulted from the contraction of W_i. Clearly, $w_jy_j, w_jy_{j+1} \in E(Y_j^*)$. Since $|W_j| \geq 2$, we have $|Y_j^*| < n$.

Let us now define an x_i-x_{i+1} path P_i in X_i and an y_j-y_{j+1} path Q_j in Y_j for all i and j as follows:

Set $P_i := X_i$ if $|X_i| = 2$. Clearly, $\ell(P_i) = 1 \geq \alpha(t)\left(\frac{|X_i|}{27}\right)^{\beta}$. In the other case, $|X_i| \geq 3$. By Lemma 5.1 and Lemma 2.4(iii), there is an x_i-x_{i+1} path P_i in $X_i := X_i^* - u_i$ satisfying $\ell(P_i) \geq \alpha(t)\left(\frac{|X_i^*|-1}{27}\right)^{\beta} = \alpha(t)\left(\frac{|X_i|}{27}\right)^{\beta}$.

Set $Q_j := Y_j$ if $|Y_j| = 2$. Clearly, $\ell(Q_j) = 1 \geq \alpha(t)(\frac{|Y_j|}{27})^{\beta}$. In the other case, $|Y_j| \geq 3$. By Lemma 5.1 and Lemma 2.4(iii), there is a y_j-y_{j+1} path Q_j in $Y_j := Y_j^* - w_j$ satisfying $\ell(Q_j) \geq \alpha(t)\left(\frac{|Y_j^*|-1}{27}\right)^{\beta} = \alpha(t)\left(\frac{|Y_j|}{27}\right)^{\beta}$.

Finally, concatenating all these P_i, all these Q_j, and the edge e, we obtain an x-y path R through e in G such that

$$\begin{aligned}
\ell(R) &= \sum_{i=1}^{p}\ell(P_i) + \sum_{j=1}^{q}\ell(Q_j) + 1 \\
&\geq \sum_{i=1}^{p}\alpha(t)\left(\frac{|X_i|}{27}\right)^{\beta} + \sum_{j=1}^{q}\alpha(t)\left(\frac{|Y_j|}{27}\right)^{\beta} + 1 \\
&\geq \alpha(t)\left(\frac{1}{27}\left(\sum_{i=1}^{p}|X_i| + \sum_{j=1}^{q}|Y_i|\right)\right)^{\beta} + 1 \quad \text{(by Corollary 3.6)} \\
&\geq \alpha(t)\left(\frac{|G|-1}{27}\right)^{\beta} + 1 \\
&> \alpha(t)\left(\frac{|G|}{28}\right)^{\beta} + 1.
\end{aligned}$$

This completes the proof of Lemma 6.1. ■

7 Proof of Theorem 2.5(c)

In this section, we establish the induction step for proving Theorem 2.5(c).

Lemma 7.1. *Suppose $n > b^{t(t-1)}$, $t \geq 2$, and Theorem 2.5 holds for graphs with at most $n-1$ vertices and for graphs containing no $K_{3,t}$-minors. Then Theorem 2.5(c) holds for graphs with n vertices.*

Proof. To show the existence of an x-y path of length at least $\alpha(t)n^\beta$ in G, we search for it from x and proceed step by step to y. At a certain point, the remaining graph may no longer be 3-connected. In this case, we are forced to choose one out of several parts of this graph. While our choice may be "good" at some stage, it may become undesirable at certain later stage, thereby we have to come back to modify our choice. This process is very sophisticated, and the notion of "magic minor" was used in [4] to guide the direction of our search and to help us explain things in a precise and concise way. To prove the present lemma, we need a modified version of this concept.

Let H_0 be an induced subgraph of G and let x_0 and y_0 be two distinct vertices of H_0 such that $H_0 + x_0y_0$ is 2-connected. We say that (H_0, x_0, y_0) is a *magic minor* of (G, x, y) if the following conditions are satisfied:

(M1) $G - (V(H_0) - \{x_0, y_0\})$ contains two vertex-disjoint paths X_0, Y_0 from x, y to x_0, y_0, respectively;

(M2) $U_0 := G - V(H_0)$ is connected and H_0^* is 3-connected, where $H_0^* := G/U_0$ if H_0 is 2-connected and $H_0^* := (G/U_0) + x_0y_0$ otherwise;

(M3) U_0 is the disjoint union of two connected vertex subsets Λ_0 and Ω_0 such that $V(X_0) \subseteq \Lambda_0 \cup \{x_0\}$, $V(Y_0) \subseteq \Omega_0 \cup \{y_0\}$, and $N(V(H_0) - \{y_0\}) \subseteq \Lambda_0 \cup \{y_0\}$; and

(M4) $|H_0| \geq n/2$ and the inequality $\alpha(t)a^\beta + \ell(X_0) + \ell(Y_0) \geq (a + 4(n - |H_0|))^\beta$ holds for any $a \geq \frac{n}{432}$.

We also say that (H_0, x_0, y_0) is a *near-magic minor* of (G, x, y) if (M1), (M2) and (M3) hold.

Claim 7.1. *Let $\mathcal{M}$ denote the set of all magic minors of (G, x, y). Then $\mathcal{M} \neq \emptyset$.*

To justify this, let $H_0 := G - x$, let $y_0 := y$, and let x_0 be a neighbor of x other than y. Then $G - (V(H_0) - \{x_0, y_0\})$ contains two vertex-disjoint paths $X_0 := xx_0$ and $Y_0 := y_0$. So (M1) holds. Clearly, $U_0 := G - V(H_0) = \{x\}$ is connected. From the 3-connectivity of G, we see that H_0 is 2-connected and $H_0^* := G/U_0 = G$ is 3-connected. So (M2) holds. Setting $\Lambda_0 = \{x\}$ and $\Omega_0 = \emptyset$ yields (M3). Obviously, $|H_0| = n - 1 \geq n/2$. Moreover, for any $a \geq \frac{n}{432}$, we have

$$\alpha(t)a^\beta + \ell(X_0) + \ell(Y_0) = \alpha(t)a^\beta + 1 \geq \alpha(t)(a^\beta + 1) \geq \alpha(t)\left(a + (b-1)\right)^\beta \geq \alpha(t)(a+4)^\beta,$$

where the second inequality follows from Lemma 3.4. Note that $(a + 4(n - |H_0|))^\beta = (a+4)^\beta$ for $|H_0| = n - 1$, so (M4) also holds. Therefore $(H_0, x_0, y_0) \in \mathcal{M}$, as claimed.

We reserve the triple (H_0, x_0, y_0) for a magic minor in $\mathcal{M}$ with smallest $|H_0|$ hereafter. Now let us recursively define a sequence of near-magic minors of (G, x, y) starting from (H_0, x_0, y_0). (The construction of this sequence is quite complex. However, once it is understood, the remaining arguments are mostly easy consequences of this construction and previous claims.)

At a general step, suppose we have already had a near-magic minor (H_i, x_i, y_i) of (G, x, y, z) for some $i \geq 0$; that is, (m0) H_i is an induced subgraph of G and $H_i + x_iy_i$ is 2-connected;

(m1) $G - (V(H_i) - \{x_i, y_i\})$ contains two vertex-disjoint paths X_i, Y_i from x, y to x_i, y_i, respectively;

(m2) $U_i := G - V(H_i)$ is connected and H_i^* is 3-connected, where $H_i^* := G/U_i$ if H_i is 2-connected and $H_i^* := (G/U_i) + x_iy_i$ otherwise;

(m3) U_i is the disjoint union of two connected sets Λ_i and Ω_i such that $V(X_i) \subseteq \Lambda_i \cup \{x_i\}$, $V(Y_i) \subseteq \Omega_i \cup \{y_i\}$, and $N(V(H_i) - \{y_i\}) \subseteq \Lambda_i \cup \{y_i\}$;

(m4) $|H_i| \geq n/2$.

Depending on whether or not $\{x_i, y_i\}$ is a cutset of H_i, we construct the following objects according to two rules (R1) and (R2):

- $(H_{i+1}, x_{i+1}, y_{i+1})$, Λ_{i+1}, and Ω_{i+1}, where H_{i+1} is a subgraph of H_i and $x_{i+1}, y_{i+1} \in V(H_{i+1})$. Let $U_{i+1} := G - V(H_{i+1})$, let $H_{i+1}^* := G/U_{i+1}$ if H_{i+1} is 2-connected and $H_{i+1}^* := (G/U_{i+1}) + x_{i+1}y_{i+1}$ otherwise, and let u_{i+1} be the vertex of H_{i+1}^* resulted from the contraction of U_{i+1}. (U_{i+1}, u_{i+1}), and H_{i+1}^*;
- $(H_{i+1,j}, x_{i+1}, y_{i+1,j})$ for $j = 1, 2, \ldots, s_{i+1}$, where $H_{i+1,j}$ is a subgraph of H_i and $x_{i+1}, y_{i+1,j} \in V(H_{i+1,j}$. Let $U_{i+1,j} := G - V(H_{i+1,j})$, let $H_{i+1,j}^* := G/U_{i+1,j}$ if $H_{i+1,j}$ is 2-connected and $H_{i+1,j}^* := (G/U_{i+1,j}) + x_{i+1}y_{i+1,j}$ otherwise, and let $u_{i+1,j}$ be the vertex of $H_{i+1,j}^*$ resulted from the contraction of $U_{i+1,j}$; and
- $(F_{i+1,j}, x'_{i+1}, y'_{i+1,j})$ for $j = 0, 1, \ldots, t_{i+1}$, where $F_{i+1,j}$ is a subgraph of H_i and $x'_{i+1}, y'_{i+1,j} \in V(F_{i+1,j}$. Let $W_{i+1,j} := G - F_{i+1,j}$, let $F_{i+1,j}^* := G/W_{i+1,j}$ if $F_{i+1,j}$ is 2-connected and $F_{i+1,j}^* := (G/W_{i+1}) + x'_{i+1}y'_{i+1}$ otherwise, and let $w_{i+1,j}$ be the vertex of $F_{i+1,j}^*$ resulted from the contraction of $W_{i+1,j}$.

In what follows, we set $\bar{\tau}(D) := \tau(G/(G - D))$ for each subgraph D of G.

(R1) Suppose $\{x_i, y_i\}$ is a cutset of H_i. Let B_i be an $\{x_i, y_i\}$-bridge of H_i with largest size and set $H_{i+1} := G[B_i]$. Let $H_{i+1,j}$, $j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial $\{x_i, y_i\}$-bridges of H_i different from B_i, and let $x_{i+1} = x_i$, $y_{i+1} = y_i$, and $y_{i+1,j} = y_i$ for $1 \leq j \leq s_{i+1}$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. In this case $(F_{i+1,j}, x'_{i+1}, y'_{i+1,j})$, $(W_{i+1,j}, w_{i+1,j})$, and $F_{i+1,j}^*$ for $j = 0, 1, \ldots, t_{i+1}$ are all set to $\emptyset$.

(R2) Suppose $\{x_i, y_i\}$ is not a cutset of H_i. Let B_i be the unique block of $H_i - x_i$ containing y_i, and let $\bar{B}_i$ be the union of all nontrivial (x_i, B_i)-bridges of H_i, if any, and be a trivial such bridge otherwise.

If there exists some (x_i, B_i)-bridge $B_{i,x}$ in $\bar{B}_i$ with $|B_{i,x}| \geq |\bar{B}_i|/4$, we choose such $B_{i,x}$ with largest size; otherwise, there exists some (x_i, B_i)-bridge $B_{i,x}$ in $\bar{B}_i$ with $\bar{\tau}(B_{i,x}) < t$ (see Claim 7.10); we choose such $B_{i,x}$ with largest size. Let $\{z_i\} = V(B_i) \cap V(B_{i,x})$.

If there exists some (y_i, z_i)-bridge $B_{i,y}$ of B_i with $|B_{i,y}| \geq |B_i|/4$, we choose such $B_{i,y}$ with largest size; otherwise, there exists some (z_i, y_i)-bridge $B_{i,y}$ of B_i with $\bar{\tau}(B_{i,y}) < t$ (see Claim 7.10); we choose such $B_{i,y}$ with largest size.

Depending on the sizes of $B_{i,x}$ and $B_{i,y}$, we distinguish between two cases:

(1) $|B_{i,x}| \geq |B_{i,y}|$. In this case, let $H_{i+1} := G[B_{i,x}]$ and set $x_{i+1} := x_i$ and $y_{i+1} := z_i$. Let $H_{i+1,j}$, $j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial (x_i, B_i)-bridges of H_i different from H_{i+1}, and let $\{y_{i+1,j}\} := V(H_{i+1,j}) \cap V(B_i)$. Let $F_{i+1,0} = G[B_{i,y}]$ and set $x'_{i+1} := z_i$ and $y'_{i+1} := y_i$. Let $F_{i+1,j}$, $j = 1, 2, \ldots, t_{i+1}$, be all the nontrivial $\{y_i, z_i\}$-bridges of B_i different from $F_{i+1,0}$ (the only trivial $\{y_i, z_i\}$-bridge is the edge $y_i z_i$, if any), and set $x'_{i+1,j} := z_i$ and $y'_{i+1,j} = y_i$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - (V(H_{i+1}) \cup V(F_{i+1,0})))$ and $\Omega_{i+1} := \Omega_i \cup V(F_{i+1,0} - y_{i+1})$.

(2) $|B_{i,x}| < |B_{i,y}|$. In this case, let $H_{i+1} := G[B_{i,y}]$ and set $x_{i+1} := z_i$ and $y_{i+1} := y_i$. Let $H_{i+1,j}$, $j = 1, 2, \ldots, s_{i+1}$, be all the nontrivial $\{z_i, y_i\}$-bridges of B_i different from H_{i+1}, and set $y_{i+1,j} = y_i$. Let $F_{i+1,0} := G[B_{i,x}]$, let $x'_{i+1} := x_i$, and let $y'_{i+1} = z_i$. Let $F_{i+1,j}$, $j = 1, 2, \ldots, t_{i+1}$, be all the nontrivial (x_i, B_i)-bridges of H_i different from $F_{i+1,0}$, and let $\{y'_{i+1,j}\} = V(F_{i+1,j}) \cap V(B_i)$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$.

We shall verify that $(H_{i+1}, x_{i+1}, y_{i+1})$ is a near-magic minor of (H, x, y). We terminate this construction process when $|H_{i+1}| < n/2$ or when $\sum_{p=1}^{i} \sum_{j=1}^{s_i} |H_{p,j}| > (n - |H_i|)/2$.

From the construction process we see that

$$|H_i| \leq |H_{i+1}| + |\cup_{j=1}^{s_{i+1}} H_{i+1,j}| + |\cup_{j=0}^{t_{i+1}} F_{i+1,j}|. \tag{7.1}$$

Let us exhibit some additional properties enjoyed by the objects constructed above.

Claim 7.2. *The graphs U_{i+1}, $U_{i+1,j}$ and $W_{i+1,j}$ are all connected. Both H_{i+1} and $F_{i+1,0}$ are induced subgraphs of G. The graph $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. The graphs H^*_{i+1}, $H^*_{i+1,j}$ and $F^*_{i+1,j}$ are all 3-connected. $\{u_{i+1}x_{i+1}, u_{i+1}y_{i+1}\} \subseteq E(H^*_{i+1})$, $\{u_{i+1,j}x_{i+1}, u_{i+1,j}y_{i+1,j}\} \subseteq E(H^*_{i+1,j})$, and $\{w_{i+1}x'_{i+1}, w_{i+1} y'_{i+1}\} \subseteq E(F^*_{i+1,0})$. Moreover, U_{i+1} is the disjoint union of Λ_{i+1} and Ω_{i+1}, both $G[\Lambda_{i+1}]$ and $G[\Omega_{i+1}]$ are connected, and $N(V(H_{i+1}) - \{y_{i+1}\}) \subseteq \Lambda_{i+1} \cup \{y_{i+1}\}$. In particular, $G - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ contains two vertex-disjoint paths from x, y to x_{i+1}, y_{i+1}, respectively.*

Indeed, since G is 3-connected and U_i is connected (by (m2)), it follows from (R1) and (R2) that U_{i+1}, $U_{i+1,j}$, and W_{i+1} are all connected. Since H_i is an induced subgraph of G (by (m0)), from (R1) and (R2) we deduce that both H_{i+1} and $F_{i+1,0}$ are induced subgraphs of G. Since $|H_i| \geq n/2 \geq b^{t(t-1)}/2$ (by (m4)) and $H_i + x_i y_i$ is 2-connected (by (m0)), it is easy to see that $|H_{i+1}| \geq 3$ and $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. If H_{i+1} is 2-connected, then H^*_{i+1} is 3-connected by Lemma 3.3. If H_{i+1} is not 2-connected then, since $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected, $H^*_{i+1} = (G/U_{i+1}) + x_{i+1}y_{i+1} = (G + x_{i+1}y_{i+1})/U_{i+1}$ is again 3-connected. Similarly, we can show that both $F^*_{i+1,0}$ (if nonempty) and $H^*_{i+1,j}$ are 3-connected. The properties enjoyed by Λ_{i+1} and Ω_{i+1} follow instantly from (m3) and the construction of Λ_{i+1} and Ω_{i+1}. The rest of Claim 7.2 are implied by the definitions of $H^*_{i+1}, H^*_{i+1,j}$ and $F^*_{i+1,0}$ in (R1) or (R2).

The next two claims follow instantly from (R1) and (R2).

Claim 7.3. *There exist two vertex-disjoint paths in $H_i - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ from x_{i+1}, y_{i+1} to x_i, y_i, respectively. For each j with $1 \leq j \leq s_{i+1}$, there exist two vertex-disjoint paths in $H_i - (V(H_{i+1,j}) - \{x_{i+1}, y_{i+1,j}\})$ from x_{i+1}, $y_{i+1,j}$ to x_i, y_i, respectively. Moreover, for each j with $1 \leq j \leq t_{i+1}$, if $F_{i+1,j}$ is defined, then there exist two vertex-disjoint paths in $H_i - (V(F_{i+1,j}) - \{x'_{i+1}, y'_{i+1,j}\})$ from x'_{i+1}, $y'_{i+1,j}$ to x_i, y_i, respectively. Hence, by Claim 7.2, $(H_{i+1}, x_{i+1}, y_{i+1})$ is a near-magic minor of (G, x, y).*

Claim 7.4. *For (R2), either $x_{i+1} = x_i$, $y_{i+1} = z_i = x'_{i+1}$, $y'_{i+1} = y_i$; or $x_{i+1} = z_i = y'_{i+1}$, $y_{i+1} = y_i$, and $x'_{i+1} = x_i$. Moreover, $H_{i+1} \cap F_{i+1,0} = \{z_i\}$, $|V(H_{i+1,j}) \cap V(F_{i+1,0})| \le 1$, $V(H_{i+1,j}) \cap V(F_{i+1,0}) \subseteq \{y'_{i+1}, y_{i+1,j}\}$, and $H_{i+1} - \{x_{i+1}, y_{i+1}\}$ and $H_{i+1,j} - \{x_{i+1}, y_{i+1,j}\}$, for all $j = 1, 2, \ldots, s_{i+1}$, are pairwise vertex-disjoint.*

Claim 7.5. *Let D be an induced subgraph of G that is a chain of blocks $v_0 D_1 v_1 D_2 v_2 \ldots v_m D_m v_{m+1}$. Suppose $G - D$ and $G - D_k$ are connected for all $k = 1, 2, \ldots, m$. Then $\bar{\tau}(D) = \max_{1 \le k \le m} \bar{\tau}(D_k)$.*

It is obvious that any $K_{3,p}$-minor in $G/(G - D_k)$ yields a $K_{3,p}$-minor in $G/(G - D)$. So $\bar{\tau}(D) \ge \max_{1 \le k \le m} \bar{\tau}(D_k)$.

Conversely, suppose $V_1, V_2, \ldots, V_{p+3}$ form a representation of a $K_{3,p}$-minor in $G/(G - D)$. Let u denote the vertex resulted from the contraction of $G - D$.

If $u \notin V_i$ for any $i \in \{1, 2, \ldots, p+3\}$ then, by 3-connectedness of $K_{3,p}$, there exists $k \in \{1, 2, \ldots, m\}$ such that $V'_j := V_j \cap V(D_k)$ are all connected. Clearly, $V'_1, V'_2, \ldots, V'_{p+3}$ form a representation of a $K_{3,p}$-minor in $G/(G - D_k)$. So $\bar{\tau}(D) \le \max_{1 \le k \le m} \bar{\tau}(D_k)$.

If $u \in V_i$ for some $i \in \{1, 2, \ldots, p+3\}$ then, by 3-connectedness of $K_{3,p}$, there exists $k \in \{1, 2, \ldots, m\} - \{i\}$ such that $V'_j := V_j \cap V(D_k)$, with $j \ne i$, are all connected. Let $V'_i := V(G) - V(D_k)$. Clearly, $V'_1, V'_2, \ldots, V'_{p+3}$ form a representation of a $K_{3,p}$-minor in $G/(G - D_k)$. So $\bar{\tau}(D) \le \max_{1 \le k \le m} \bar{\tau}(D_k)$. Thus the claim is justified.

Claim 7.6. *If H^*_{i+1} is not a minor of G, then $s_{i+1} = 0$ (which means that there is no $H_{i+1,j}$). All $H^*_{i+1,j}$ and all $F^*_{i+1,j}$ with $j \ge 1$ are minors of G. If $F^*_{i+1,0}$ is not a minor of G, then $t_{i+1} = 0$. There exists a path P_{i+1} (resp. $P_{i+1,j}$, and $R_{i+1,j}$) in H_{i+1} (resp. $H_{i+1,j}$, and $F_{i+1,j}$) connecting x_{i+1} and y_{i+1} (resp. x_{i+1} and $y_{i+1,j}$, and x'_{i+1} and $y'_{i+1,j}$) such that*

- *$\ell(P_{i+1}) \ge \alpha(\bar{\tau}(H_{i+1}))(|H_{i+1}|/27)^\beta$,*
- *$\ell(P_{i+1,j}) \ge \alpha(\bar{\tau}(H_{i+1,j}))(|H_{i+1,j}|/27)^\beta$, and*
- *$\ell(R_{i+1,j}) \ge \alpha(\bar{\tau}(F_{i+1,j}))(|F_{i+1,j}|/27)^\beta$.*

Clearly, H^*_{i+1} is a minor of G if $H^*_{i+1} = G/U_{i+1}$. It remains to consider the case when $H^*_{i+1} = G/U_{i+1} + x_{i+1}y_{i+1}$. If $s_{i+1} > 0$ then $H_{i+1,1}$ exists; in this case, H^*_{i+1} can be obtained from G by contracting to y_{i+1} the graph $H_{i+1,1} - \{x_{i+1}\}$, and by contracting $U_{i+1} - (V(H_{i+1,1}) - \{x_{i+1}, y_{i+1}\})$ (which is connected since it contains U_i and all $H_{i+1,j}$ and $F_{i+1,j}$ (if nonempty) have neighbors in U_i). So H^*_{i+1} is again a minor of G.

Similarly, if $F^*_{i+1,0}$ is not a minor of G, then $t_{i+1} = 0$; and all $H^*_{i+1,j}$ and all $F^*_{i+1,j}$ with $j \ge 1$ are minors of G.

To show the existence of the desired path P_{i+1}, note that H_{i+1} is a chain of blocks $v_0 D_0 v_1 D_1 v_2 \ldots v_m D_m v_{m+1}$, with $v_0 = x_{i+1}$ and $v_{m+1} = y_{i+1}$. By Lemma 3.3, $G/(G - D_k)$ is either 3-connected or a triangle for $k = 0, 1, \ldots, m$. In the former case Lemma 5.1 guarantees the existence of a u_k-u_{k+1} path R_k in D_k with $\ell(R_k) \ge \alpha(\bar{\tau}(D_k))\,(\delta(t, D_k)|D_k|)^\beta \ge \alpha(\bar{\tau}(H_{i+1}))\,(|D_k|/27)^\beta$, where the first inequality follows from Lemma 7.5 and the second from Lemma 2.4(iii); in the latter case this statement holds trivially. Concatenating all these R_k, we obtain an x_{i+1}-y_{i+1} path P_{i+1} in H_{i+1} with

$$\ell(P_{i+1}) \ge \alpha(\bar{\tau}(H_{i+1})) \sum_{k=0}^{m} (|D_k|/27)^\beta \ge \alpha(\bar{\tau}(H_{i+1})) \left(\sum_{k=0}^{m} |D_k|/27 \right)^\beta \ge \alpha(\bar{\tau}(H_{i+1}))(|H_{i+1}|/27)^\beta,$$

where the second inequality follows from Corollary 3.6.

The existence of $P_{i+1,j}$ and $R_{i+1,j}$ can be justified likewise. This establishes the claim.

Claim 7.7. *(i)* $\tau(G) \geq \lceil \bar{\tau}(H_{i+1})/3 \rceil + \sum_{j=1}^{s_{i+1}} \lceil \bar{\tau}(H_{i+1,j})/3 \rceil$ *and (ii)* $\tau(G) \geq \sum_{j=0}^{t_{i+1}} \lceil \bar{\tau}(F_{i+1,j})/3 \rceil$.

We only prove the first inequality as the second one can be established similarly.

Let us first show that H^*_{i+1} contains a $K_{3,\lceil \bar{\tau}(H_{i+1})/3 \rceil}$-minor Σ_{i+1} rooted at $\{x_{i+1}, y_{i+1}, u_{i+1}\}$. For this purpose, note that H_{i+1} is a chain of blocks $v_0 D_0 v_1 D_1 v_2 \ldots v_m D_m v_{m+1}$, with $v_0 = x_{i+1}$ and $v_{m+1} = y_{i+1}$. Statement (i) holds trivially if $G[H_{i+1}]$ is a path (which implies $\bar{\tau}(H_{i+1}) = 1$). Thus, by Claim 7.5, we may assume the existence of a nontrivial block D_k of H_{i+1} such that $\bar{\tau}(H_{i+1}) = \tau(G/(G - D_k))$. In view of Lemma 2.1, $G/(G - D_k)$ contains a $K_{3,\lceil \bar{\tau}(H_{i+1})/3 \rceil}$-minor rooted at v_k, v_{k+1}, and the vertex resulted from contracting $G - V(D_k)$. Clearly, this minor leads to a $K_{3,\lceil \bar{\tau}(H_{i+1})/3 \rceil}$-minor Σ_{i+1} of H^*_{i+1} rooted at $\{x_{i+1}, y_{i+1}, u_{i+1}\}$, as desired.

Similarly, $H^*_{i+1,j}$ contains a $K_{3,\lceil \bar{\tau}(H_{i+1,j})/3 \rceil}$-minor $\Sigma_{i+1,j}$ rooted at $\{x_{i+1}, y_{i+1,j}, u_{i+1,j}\}$. By (R1) and (R2), the neighbors of $H_{i+1} - \{x_{i+1}, y_{i+1}\}$ and those of $H_{i+1,j} - \{x_{i+1}, y_{i+1,j}\}$, for $1 \leq j \leq s_{i+1}$, are all in $U_i = G - V(H_i)$ if H_{i+1} is defined in (R1) or $H_{i+1} = B_{i,x}$ in (R2), and all in $U_i = G - V(H_i)$ and $\bigcup_{k=0}^{t_{i+1}} (F_{i+1,k} - z_i)$ otherwise. By (m2), U_i is connected. So we can merge the above Σ_{i+1} and $\Sigma_{i+1,j}$ to form a $K_{3,p}$-minor of G with $p \geq \lceil \bar{\tau}(H_{i+1})/3 \rceil + \sum_{j=1}^{s_{i+1}} \lceil \bar{\tau}(H_{i+1,j})/3 \rceil$. Hence (i) follows.

Clearly, $1 \leq \bar{\tau}(H_{i+1,j}) \leq \tau(G) \leq t$ for $1 \leq j \leq s_{i+1}$. Similarly, $1 \leq \bar{\tau}(F_{i+1,j}) \leq \tau(G) \leq t$ for $0 \leq j \leq t_{i+1}$. From Claim 7.7 we see that

Claim 7.8. $s_{i+1} \leq t - 1$ *and* $t_{i+1} \leq t - 1$.

Set $H_{i+1,0} = H_{i+1}$. Throughout the remainder of our proof, s^*_{i+1} stands for the number of bridges $H_{i+1,j}$, with $0 \leq j \leq s_{i+1}$, satisfying $\bar{\tau}(H_{i+1,j}) = t$, and t^*_{i+1} stands for the number of bridges $F_{i+1,j}$, with $0 \leq j \leq t_{i+1}$, satisfying $\bar{\tau}(F_{i+1,j}) = t$.

Claim 7.9. $s^*_{i+1} \leq 3$ *and equality holds only if* $s_{i+1} = 2$*;* $t^*_{i+1} \leq 3$ *and equality holds only if* $t_{i+1} = 2$.

From Claim 7.7(i) we deduce that $t \geq s^*_{i+1} \lceil t/3 \rceil + (s_{i+1} + 1 - s^*_{i+1})$, so the first part of our claim follows. The second part can be justified likewise.

Claim 7.10. $B_{i,x}$ *and* $B_{i,y}$ *in (R2) are well defined. Moreover,* $|B_{i,x}| \geq |\bar{B}_i|/(2t)$ *and* $|B_{i,y}| \geq |B_i|/(2t)$.

We only prove the statements for $B_{i,x}$ as the proof for $B_{i,y}$ goes along the same line.

We may assume that all (x_i, B_i)-bridges B in $\bar{B}_i$ satisfy $|B| < |\bar{B}_i|/4$, for otherwise, according to (R2) we choose $B_{i,x}$ to be one with $|B_{i,x}|$ maximum. This implies $|B_{i,x}| \geq |\bar{B}_i|/4 \geq |\bar{B}_i|/(2t)$ (as $t \geq 2$).

So there is an (x_i, B_i)-bridges B in $\bar{B}_i$ with $\bar{\tau}(B) < t$, for otherwise, all such B satisfy $\bar{\tau}(B) = t$. Since $s^*_{i+1} \leq 3$ (by Claim 7.9), there exists an (x_i, B_i)-bridge B in $\bar{B}_i$ such that $|B| \geq |\bar{B}_i|/3 \geq |\bar{B}_i|/4$, a contradiction.

By Claim 7.9, $s^*_{i+1} \leq 2$; that is, the number of (x_i, B_i)-bridges B in $\bar{B}_i$ with $\bar{\tau}(B) = t$ is at most two. For such B, the definition of $B_{i,x}$ (see (R2)) implies that $|B| < |\bar{B}_i|/4$. Hence, using Claim 7.8, we get $|B_{i,x}| \geq (|\bar{B}_i| - 2|\bar{B}_i|/4)/t = |\bar{B}_i|/(2t)$, as desired.

Claim 7.11. *We may assume that the following three statements hold:*

(i) *if* $\bar{\tau}(H_{i+1}) < t$, *then* $|H_{i+1}| < |H_i|/(8t^2)$,

(ii) *if* $\bar{\tau}(H_{i+1,j}) < t$, *then* $|H_{i+1,j}| < |H_i|/(8t^2)$, *and*

(iii) *if* $\bar{\tau}(F_{i+1,j}) < t$, *then* $|F_{i+1,j}| < |H_i|/(8t^2)$.

We prove (i) only since the other two statements can be established similarly.

Suppose $\bar{\tau}(H_{i+1}) < t$ and $|H_{i+1}| \geq |H_i|/(8t^2)$. Then $|H_{i+1}| \geq n/(16t^2)$ by (m4). Hence the x_{i+1}-y_{i+1} path exhibited in Claim 7.6 has length at least

$$\alpha(t-1)\left(\frac{|H_{i+1}|}{27}\right)^{\beta} \geq \alpha(t-1)\left(\frac{n}{432t}\right)^{\beta} = \alpha(t)\left(\frac{b^{2(t-1)}n}{432t^2}\right)^{\beta} \geq \alpha(t)n^{\beta},$$

where the equality follows from (5.1). (Observe that in the last inequality, we need $t \geq 2$; and when $t = 2$ we need $b = 1729$ as $432t^2 = 1728 = b - 1$.) Clearly, P_{i+1} can be extended to an x-y path in G with length at least $\alpha(t)n^{\beta}$.

Claim 7.12. (i) $\bar{\tau}(H_{i+1}) = t$, *and* (ii) $|H_{i+1}| \geq |H_{i+1,j}|$ *for* $j = 1, 2, \ldots, s_{i+1}$.

If (R1) applies, then $|H_{i+1}| \geq |H_i|/t$. If (R2) applies, then $|H_{i+1}| = \max\{|B_{i,x}|, |B_{i,y}|\} \geq (|B_{i,x}| + |B_{i,y}|)/2 \geq |H_i|/(2t)$ by Claim 7.10. It follows from Claim 7.11(i) that $\bar{\tau}(H_{i+1}) = t$. By (i) and (R2), we get (ii) immediately.

Claim 7.13. $|H_{i+1}| \geq |F_{i+1,j}|$ *for* $j = 0, 1, \ldots, t_{i+1}$.

To justify this, recall (R2); we only prove the statement for the case when $|B_{i,x}| \geq |B_{i,y}|$ as the proof for the other case goes along the same line.

If $|B_{i,y}| \geq |B_i|/4$ then according to (R2), $|F_{i+1,0}| \geq |F_{i+1,j}|$ for $j = 1, 2, \ldots, t_{i+1}$, and hence $|H_{i+1}| = |B_{i,x}| \geq |B_{i,y}| = |F_{i+1,0}|$. So we assume that $|B_{i,y}| < |B_i|/4$. Then, by (R2), $\bar{\tau}(F_{i+1,0}) < t$ and $|F_{i+1,j}| < |B_i|/4$ for $0 \leq j \leq t_{i+1}$. By Claim 7.9, we have $t^*_{i+1} \leq 3$. From Claim 7.8 and Claim 7.11(iii), it follows that $|B_i| \leq \sum_{j=0}^{t_{i+1}} |F_{i+1,j}| \leq t^*_{i+1}|B_i|/4 + t|H_i|/(8t^2) \leq 3|B_i|/4 + t|H_i|/(8t^2)$, implying $|B_i| \leq |H_i|/(2t)$. Hence, by Claim 7.8 and Claim 7.12(ii), $|H_{i+1}| \geq |\bar{B}_i|/t \geq |H_i - B_i|/t \geq (1 - 1/(2t))|H_i|/t \geq |H_i|/(2t) \geq |B_i|$, which yields the statement as desired.

Claim 7.14. (i) $|H_{i+1}| \geq |H_i|/8 \geq n/16$, (ii) $|H_{i+1}| \geq |\cup_{j=0}^{t_{i+1}} F_{i+1,j}|/4$, *and* (iii) $|F_{i+1,0}| \geq |\cup_{j=0}^{t_{i+1}} F_{i+1,j}|/4$ *if* $\bar{\tau}(F_{i+1,0}) = t$.

To justify (i), we appeal to inequality (7.1). In view of Claim 7.8, Claim 7.11, Claim 7.12(ii), and Claim 7.13, we obtain

$$|H_i| \leq (s^*_{i+1} + t^*_{i+1})|H_{i+1}| + 2t\frac{|H_i|}{8t^2} \leq 6|H_{i+1}| + \frac{|H_i|}{4t},$$

where the second inequality follows from Claim 7.9. So $|H_{i+1}| \geq (4t-1)|H_i|/(24t) \geq |H_i|/8$ as $t \geq 2$. By (m4), $|H_i| \geq n/2$. Hence inequality (i) is established.

By Claim 7.9, $t^*_{i+1} \leq 3$. From Claim 7.11(iii) we see that $|\bigcup_{j=0}^{t_{i+1}} F_{i+1,j}| \leq 3|H_{i+1}| + t|H_i|/(8t^2) \leq 3|H_{i+1}| + |H_{i+1}|/t \leq 4|H_{i+1}|$, where the second inequality follows from (i). So inequality (ii) is also proved.

Inequality (iii) follows instantly from the construction rule (R2).

Claim 7.15. *Let P_{i+1} and $R_{i+1} := R_{i+1,0}$ be the paths as described in Claim 7.6. Then*

(i) $\ell(P_{i+1}) \geq \alpha(t)(|H_i|/216)^\beta$ *and*

(ii) $\ell(R_{i+1}) \geq \alpha(t)(|\bigcup_{j=0}^{t_{i+1}} F_{i+1,j}|/108)^\beta$.

As in Claim 7.6, $\ell(P_{i+1}) \geq \alpha(\bar{\tau}(H_{i+1}))(|H_{i+1}|/27)^\beta$. By Claim 7.12 and Claim 7.14(i), we have $\bar{\tau}(H_{i+1}) = t$ and $|H_{i+1}| \geq |H_i|/8$. So (i) follows instantly.

By Claim 7.6, $\ell(R_{i+1}) \geq \alpha(\bar{\tau}(F_{i+1,0}))(|F_{i+1,0}|/27)^\beta$. Clearly, $\bar{\tau}(F_{i+1,0}) \leq \tau(G) \leq t$. If $\bar{\tau}(F_{i+1,0}) = t$, then (ii) follows from Claim 7.14(iii). So we assume that $\bar{\tau}(F_{i+1,0}) \leq t-1$. Thus

$$\ell(R_{i+1}) \geq \alpha(t-1)(|F_{i+1,0}|/27)^\beta = \alpha(t)\left(b^{2(t-1)}|F_{i+1,0}|/27\right)^\beta,$$

where the equality follows from (5.1). By Claim 7.10, $|F_{i+1,0}| \geq |\bigcup_{j=0}^{t_{i+1}} F_{i+1,j}|/(2t)$. Plugging this into the above inequality, we get (ii). So the claim is justified.

Suppose $\{(H_i, x_i, y_i) : i = 0, 1, \ldots, k\}$ is a maximal sequence of near-magic minors recursively constructed from (H_0, x_0, y_0) according to (R1) and (R2), subject to the following two constraints:

(S1) $|H_k| \geq \frac{n}{2}$, and

(S2) for each s with $1 \leq s \leq k$,

$$\sum_{i=1}^{s}\sum_{j=1}^{s_i}|H_{i,j}| \leq \frac{1}{2}(n - |H_s|).$$

Starting from (H_k, x_k, y_k) and using (R1) and (R2), we can still construct

- $(H_{k+1}, x_{k+1}, y_{k+1})$ and $(H_{k+1,j}, x_{k+1}, y_{k+1,j})$ for $j = 1, 2, \ldots, s_{k+1}$,
- $(F_{k+1,j}, x'_{k+1,j}, y'_{k+1,j})$ for $j = 1, 2, \ldots, t_{k+1}$, and
- U_{k+1}, W_{k+1}, H^*_{k+1}, F^*_{k+1}, u_{k+1}, w_{k+1}, Λ_{k+1}, and Ω_{k+1}.

Since (H_0, x_0, y_0) is a magic minor of (G, x, y), from Claim 7.2 and Claim 7.3 we deduce that (H_i, x_i, y_i) is a near-magic minor of (G, x, y) for $1 \leq i \leq k+1$. Thereby, Claim 7.2 – Claim 7.15 hold for $i = 1, 2, \ldots, k$.

Note that, for $s = 1, 2, \ldots, k+1$, the vertices of G outside H_s is either outside H_0, or in $H_{i,j}$ for some pair i, j with $1 \leq i \leq s$ and $1 \leq j \leq s_i$, or in $F_{i,j}$ for some pair i, j with $1 \leq i \leq s$ and $0 \leq j \leq t_i$. Since $n - |H_s|$ is the number of vertices of G outside H_s, and $n - |H_0|$ is the number of vertices of G outside H_0, we have

Claim 7.16. *For any s with $1 \leq s \leq k+1$,*

$$|\bigcup_{i=1}^{s}\bigcup_{j=1}^{s_i} H_{i,j}| + |\bigcup_{i=1}^{s}\bigcup_{j=0}^{t_i} F_{i,j}| + (n - |H_0|) \geq n - |H_s|.$$

By Claim 7.3, $(H_{k+1}, x_{k+1}, y_{k+1})$ is a near-magic minor of (G, x, y). The maximality on k implies

Claim 7.17. *Either $|H_{k+1}| < n/2$, or $|H_{k+1}| \geq n/2$ and $\sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|)$.*

By (7.1), Claim 7.11 and Claim 7.9, we have $|H_k| \leq 3|H_{k+1}| + t|H_k|/(8t^2) + |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|$; so $|H_{k+1}| + |\cup_{j=0}^{t_{k+1}} F_{k+1,j}|/3 \geq (1 - \frac{1}{8t})|H_k|/3$. Since $t \geq 2$ and $|H_k| \geq \frac{n}{2}$ by (S1), we obtain

Claim 7.18. $|H_{k+1}| + |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|/3 \geq |H_k|/4 \geq n/8$.

Since $n \geq |\bigcup_{j=0}^{t_i} F_{i,j}|$ for all i, from Claim 7.18 the following statement follows.

Claim 7.19. $\frac{|H_{k+1}|}{27} + 4|\cup_{j=0}^{t_{k+1}} F_{k+1,j}| \geq \frac{1}{216}|\cup_{j=0}^{t_i} F_{i,j}|$ *for any* i *with* $1 \leq i \leq k$.

Let P_{k+1} be the x_{k+1}-y_{k+1} path in H_{k+1} and let $R_i := R_{i,0}$ be the x'_i-y'_i path in $F_{i,0}$, for $1 \leq i \leq k+1$, as exhibited in Claim 7.6 (with $R_i = \emptyset$ when $F_{i,0}$ is empty). Set $Q_{k+1} := P_{k+1}$. Then

$$\ell(Q_{k+1}) \geq \alpha(t) \left(\frac{|H_{k+1}|}{27} \right)^{\beta}. \tag{7.2}$$

By Claim 7.15(ii) (or trivially when $R_i = \emptyset$), we have

$$\ell(R_i) \geq \alpha(t) \left(\frac{|\bigcup_{j=0}^{t_i} F_{i,j}|}{108} \right)^{\beta}. \tag{7.3}$$

In view of Claim 7.4, there exists an x_k-y_k path Q_k in H_k such that $Q_k \supseteq Q_{k+1} \cup R_{k+1}$. Note that

$$\begin{aligned}
\ell(Q_k) &\geq \ell(Q_{k+1}) + \ell(R_{k+1}) \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} \right)^{\beta} + \alpha(t) \left(\frac{|\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}|}{108} \right)^{\beta} \quad \text{(by (7.2) and (7.3))} \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + \frac{b-1}{108} |\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}| \right)^{\beta} \quad \text{(by Claim 7.14(ii) and Lemma 3.4)} \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\cup_{j=0}^{t_{k+1}} F_{k+1,j}| \right)^{\beta} \quad \text{(for } b-1 = 1728).
\end{aligned}$$

Similarly, let Q_{k-1} be an x_{k-1}-y_{k-1} path in H_{k-1} such that $Q_{k-1} \supseteq Q_k \cup R_k$. Then

$$\begin{aligned}
\ell(Q_{k-1}) &\geq \ell(Q_k) + \ell(R_k) \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\cup_{j=0}^{t_{k+1}} F_{k+1,j}| \right)^{\beta} + \alpha(t) \left(\frac{|\bigcup_{j=0}^{t_k} F_{k,j}|}{108} \right)^{\beta} \quad \text{(by (7.3))} \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\bigcup_{j=0}^{t_{k+1}} F_{k+1,j}| + \frac{b-1}{216} |\cup_{j=0}^{t_k} F_{k,j}| \right)^{\beta} \quad \text{(by Claim 7.19 and Lemma 3.4)} \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\cup_{i=k}^{k+1} \cup_{j=0}^{t_i} F_{i,j}| \right)^{\beta} \quad \text{(for } b-1 = 1728).
\end{aligned}$$

Using Claim 7.19 and continuing in this fashion, we obtain an x_0-y_0 path Q_0 in H_0 such that $Q_0 \supseteq Q_{k+1} \cup (\bigcup_{i=1}^{k+1} R_i)$ and that

$$\ell(Q_0) \geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\bigcup_{i=1}^{k+1} \cup_{j=0}^{t_i} F_{i,j}| \right)^{\beta}.$$

Let $P := X_0 \cup Q_0 \cup Y_0$ (see (M1) for the definitions of X_0 and Y_0). Then P is an x-y path in G with

$$\begin{aligned}
\ell(P) &= \ell(Q_0) + \ell(X_0) + \ell(Y_0) \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j}| \right)^{\beta} + \ell(X_0) + \ell(Y_0) \\
&\geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j}| + 4(n - |H_0|) \right)^{\beta},
\end{aligned}$$

where the last inequality follows from (M4) because, by Claim 7.14(i), $\frac{|H_{k+1}|}{27} \geq \frac{n}{432}$.

Claim 7.20. *We may assume that* $\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|)$.

Suppose, on the contrary, that $\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| \leq \frac{1}{2}(n - |H_{k+1}|)$. Then, by Claim 7.17, we have $|H_{k+1}| < n/2$. From Claim 7.16 it can be seen that

$$|\bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j}| + (n - |H_0|) \geq n - |H_{k+1}| - \sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| \geq \frac{1}{2}(n - |H_{k+1}|).$$

Hence

$$\ell(P) \geq \alpha(t) \left(\frac{|H_{k+1}|}{27} + 4|\bigcup_{i=1}^{k+1} \bigcup_{j=0}^{t_i} F_{i,j}| + 4(n - |H_0|) \right)^{\beta} \geq \alpha(t)(2(n - |H_{k+1}|))^{\beta} \geq \alpha(t) n^{\beta}.$$

In view of Claim 7.20, we use p to denote the smallest integer i, with $0 \leq i \leq k$, such that $H_{i+1,1}$ exists. Set $H := H_p$ if (1) of (R2), with $i = p$, occurs; and set $H := B_p$ if (R1) or (2) of (R2) occurs for $i = p$. We call B_i *light* if (1) of (R2) occurs for i. Let $\tilde{H}$ be the graph obtained from G by first contracting each light B_i to a single vertex for $i = p, p+1, \ldots, k$ (in that order), then contracting $H_{k+1} - x_{k+1}$ to y_{k+1}, and finally contracting $V(G) - V(H)$ to a single vertex $\tilde{w}$.

Claim 7.21. *The following statements hold for* $\tilde{H}$*:*

(i) All the vertices y_i *and* $y_{i,j}$*, for* $p+1 \leq i \leq k+1$ *and* $1 \leq j \leq s_i$*, are contracted into* y_p *in* $\tilde{H}$*;*

(ii) All $H_{i,j}$ *remain intact in* $\tilde{H}$*, for* $p+1 \leq i \leq k+1$ *and* $1 \leq j \leq s_i$*;*

(iii) $\tilde{H}$ *is 3-connected.*

To justify the claim, for each q with $p \le q \le k$, let D_{q+1} be the graph obtained from H by contracting light B_i to a single vertex for $i = p, p+1, \dots, q$ (in that order). Set $\tilde{x} := x_p$ if $H = H_p$, and $\tilde{x} := x_{p+1}$ if $H = B_p$; and set $\tilde{y} := y_p$. Since G is 3-connected and $H_{p+1,1}$ exists, from the contraction process of light B_i's, we deduce by induction on q (using (R1) and (R2)) that, for $q = p, p+1, \dots, k$,

(1) all the vertices y_i and $y_{i,j}$, for $p+1 \le i \le q+1$ and $1 \le j \le s_i$, are contracted into $\tilde{y}$ in D_{q+1};

(2) all $H_{i,j}$ remain intact in D_{q+1}, for $p+1 \le i \le q+1$ and $1 \le j \le s_i$;

(3) D_{q+1} is 2-connected;

(4) if $\{u, v\}$ is a cutset of D_{q+1} and C is a component of $D_{q+1} - \{u, v\}$ containing neither $\tilde{x}$ nor $\tilde{y}$, then C is adjacent to $V(G) - V(H)$;

(5) if A is a cutset of D_{q+1} with A contained in $H_{q+1} - x_{q+1}$, then all A-bridges in D_{q+1} are induced subgraphs of H_{q+1}, except one which contains both $\tilde{x}$ and $\tilde{y}$; and

(6) there is at least one edge between $V(D_{q+1}) - (V(H_{q+1}) \cup \{\tilde{x}, \tilde{y}\})$ and $V(G) - V(H)$. (Indeed, if $H = H_p$, then $H_{p+1,1} - \{x_{p+1}, y_{p+1,1}\}$ is adjacent to $V(G) - V(H)$; if $H = B_p$, with $i = p$, occurs, then $F_{p+1,0} - \{x'_{p+1}, y'_{p+1,0}\}$ is adjacent to $V(G) - V(H)$.)

Clearly, (i) follows instantly from (1) and (ii) from (2) with $q = k$. Let $\tilde{D}$ be the graph obtained from D_{k+1} by contracting $H_{k+1} - x_{k+1}$ to y_{k+1}. Then $\tilde{D}$ is 2-connected by (3) and (5). Since $\tilde{H}$ is obtained from $\tilde{D}$ by adding $\tilde{w}$ (and edges from $\tilde{D}$ to $V(G) - V(H)$), from (4) and (6) we conclude that $\tilde{H}$ is 3-connected. So (iii) also holds.

Since $\tilde{H}$ is a 3-connected minor of G and both $\tilde{y}\tilde{w}$ and $\tilde{y}x_{k+1}$ are edges in $\tilde{H}$, the induction hypothesis of Theorem 2.5(a) guarantees the existence of a $\tilde{w}$-x_{k+1} path $\tilde{Q}$ in $\tilde{H} - \tilde{y}$ with

$$\begin{aligned}
\ell(\tilde{Q}) &\ge \alpha(t)\left(\frac{|\tilde{H}|}{27}\right)^{\beta} \\
&\ge \alpha(t)\left(\frac{\sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j} - \{x_i, y_{i,j}\}|}{27}\right)^{\beta} \quad \text{(by Claim 7.21(ii))} \\
&\ge \alpha(t)\left(\frac{\frac{1}{3}\sum_{i=1}^{k+1}\sum_{j=1}^{s_i}|H_{i,j}|}{27}\right)^{\beta} \\
&\ge \alpha(t)\left(\frac{1}{162}(n - |H_{k+1}|)\right)^{\beta} \quad \text{(by Claim 7.20).}
\end{aligned}$$

By Claim 7.2, U_p is the disjoint union of Λ_p and Ω_p, both $G[\Lambda_p]$ and $G[\Omega_p]$ are connected, and $N(V(H_p) - \{y_p\}) \subseteq \Lambda_p \cup \{y_p\}$. Therefore, using Claim 7.21(i), $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ contains two vertex-disjoint paths X_{k+1} and Y_{k+1} from x, y to x_{k+1}, y_{k+1}, respectively, such that

- $\tilde{Q} - \tilde{w} \subseteq X_{k+1}$, and $V(X_{k+1}) - V(\tilde{Q} - \tilde{w})$ is contained in Λ_p if $H = H_p$ and in Λ_{p+1} if $H = B_p$;
- $V(Y_{k+1}) \cap U_p$ is contained in Ω_p, and $V(Y_{k+1}) - U_p$ is contained in the union of $F_{i+1,0}$ for all i with B_i light.

Observe that $\ell(X_{k+1}) \geq \ell(\tilde{Q}) \geq \alpha(t)\left(\frac{1}{162}(n - |H_{k+1}|)\right)^{\beta}$.

Let us consider the triple $(H_{k+1}, x_{k+1}, y_{k+1})$. Since X_{k+1} and Y_{k+1} are vertex-disjoint paths in $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ from x, y to x_{k+1}, y_{k+1}, respectively, (M1) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. By Claims 7.2 and 7.3, (M2) and (M3) also hold for $(H_{k+1}, x_{k+1}, y_{k+1})$. From the maximality on k, we deduce that $(H_{k+1}, x_{k+1}, y_{k+1})$ is not a magic minor of (G, x, y) and hence it does not satisfy (M4).

Note that for any $a \geq \frac{n}{432}$, we have $a \geq \frac{1}{2} \cdot \frac{n-|H_{k+1}|}{216}$. So

$$\begin{aligned} & \alpha(t)a^{\beta} + \ell(X_{k+1}) + \ell(Y_{k+1}) \\ \geq\ & \alpha(t)\left(a^{\beta} + \left(\frac{1}{162}(n - |H_{k+1}|)\right)^{\beta}\right) \\ >\ & \alpha(t)\left(a + \frac{b-1}{432}(n - |H_{k+1}|)\right)^{\beta} \quad \text{(by Lemma 3.4)} \\ =\ & \alpha(t)\left(a + 4(n - |H_{k+1}|)\right)^{\beta} \quad (\text{for } b - 1 = 1728). \end{aligned}$$

It follows that

Claim 7.22. $|H_{k+1}| < \frac{n}{2}$.

Now we are ready to present the last part of our proof. Let Q_{k+1} be the x_{k+1}-y_{k+1} path in H_{k+1} as exhibited in (7.2). Set $Q := X_{k+1} \cup Q_{k+1} \cup Y_{k+1}$. Then Q is an x-y path in G with

$$\ell(Q) \geq \ell(Q_{k+1}) + \ell(X_{k+1}) \geq \alpha(t)\left(\left(\frac{|H_{k+1}|}{27}\right)^{\beta} + \left(\frac{1}{162}(n - |H_{k+1}|)\right)^{\beta}\right).$$

If $\frac{|H_{k+1}|}{27} \geq \frac{1}{162}(n - |H_{k+1}|)$, then

$$\ell(Q) \geq \alpha(t)\left(\frac{|H_{k+1}|}{27} + \frac{b-1}{162}(n - |H_{k+1}|)\right)^{\beta} \geq \alpha(t)n^{\beta}$$

for $(b-1)/162 \geq 2$ and $n - |H_{k+1}| \geq n/2$. If $\frac{|H_{k+1}|}{27} < \frac{1}{162}(n - |H_{k+1}|)$, then

$$\ell(Q) \geq \alpha(t)\left(\frac{b-1}{27}|H_{k+1}| + \frac{1}{162}(n - |H_{k+1}|)\right)^{\beta} \geq \alpha(t)n^{\beta}$$

for $|H_{k+1}| \geq n/16$ by Claim 7.14 and $b = 1729$. This completes the proof of Lemma 7.1 and hence of Theorem 2.5. ∎

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