# Better bounds for $k$-partitions of graphs

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**Abstract**

Let $G$ be a graph with $m$ edges, and let $k$ be a positive integer. We show that $V(G)$ admits a $k$-partition $V_1, \ldots, V_k$ such that $e(V_i) \leq \frac{1}{k^2}m + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2)$ for $i \in \{1, 2, \ldots, k\}$, and $e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m+1/4} + O(k)$, where $e(V_i)$ denotes the number of edges with both ends in $V_i$ and $e(V_1, \ldots, V_k) = m - \sum_{i=1}^{k} e(V_i)$. This answers a problem of Bollobás and Scott [2] in the affirmative. Moreover, $\binom{k+1}{2}e(V_i) + \frac{k}{2}\sum_{j \neq i} e(V_j) \leq m + O(k^2)$ for $i \in \{1, 2, \ldots, k\}$, which is close to being best possible and settles another problem of Bollobás and Scott [2].

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## 1 Introduction

The problem of finding a maximum bipartite subgraph in any graph is NP-complete. On the other hand, it is easy to show that every graph with $m$ edges contains a bipartite subgraph with at least $m/2$ edges. Edwards [4, 5] improved this lower bound to $m/2 + h(m)/4$; where here and throughout

$$h(m) = \sqrt{2m + \frac{1}{4}} - \frac{1}{2}.$$

The complete graphs $K_{2n+1}$ show that this bound is best possible.

In [3] (also see [2]), Bollobás and Scott extend Edwards' bound to $k$-partitions of graphs and prove that the vertex set of any graph with $m$ edges can be partitioned into $V_1, \ldots, V_k$ such that

$$e(V_1, \ldots, V_k) := \sum_{1 \leq i < j \leq k} e(V_i, V_j) \geq \frac{k-1}{k} m + \frac{k-1}{2k} h(m) + O(k), \tag{1.1}$$

where $e(V_i, V_j)$ is the number of edges with one end in $V_i$ and the other in $V_j$. The inequality in (1.1) holds with equality when $G$ is the complete graph of order $kn + 1$, and the $O(k)$ term is determined in [3] as $(-k^2 + 4k - 4)/(8k)$.

In many situations, one often needs to find a partition of a graph that optimizes several quantities simultaneously; see [2,6] for extensive discussions of such problems which are usually called *Judicious Partitioning Problems*. For example, Bollobás and Scott [1] showed that for any integer $k \geq 1$ and any graph $G$ of size $m$, $V(G)$ admits a partition $V_1, \ldots, V_k$ such that for $i \in \{1, 2, \ldots, k\}$,

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2} h(m). \tag{1.2}$$

The complete graphs $K_{kn+1}$ are the only extremal graphs (modulo isolated vertices).

Another example is the following result of Porter [7]: If $k$ is a power of 2 then every graph $G$ with $m$ edges has a partition of $V(G)$ into $V_1, \ldots, V_k$ such that

$$e(V_1, \ldots, V_k) \geq \frac{k-1}{k} m$$

and for $i \in \{1, \ldots, k\}$,

$$e(V_i) \leq \frac{m}{k^2} + \sqrt{\frac{m}{k}}.$$

Porter's result was improved by Bollobás and Scott [1] to $e(V_i) \leq m/k^2 + (k-1)h(m)/(2k^2)$.

When $k = 2$, Bollobás and Scott [1] obtained the following stronger result, where $N(x)$ denotes the neighborhood of the vertex $x$ in a graph.

**Theorem 1.1** *(Bollobás and Scott [1]) Let $G$ be a graph with $m$ edges. Then there is partition $V_1, V_2$ of $V(G)$ such that*

(1) $|N(x) \cap V_2| \geq |N(x) \cap V_1|$ *for all* $x \in V_1$,

(2) $e(V_i) \leq \frac{m}{4} + \frac{1}{8} h(m)$ *for* $i = 1, 2$, *and*

(3) $e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}h(m)$.

Condition (1) is essential in the proof of Theorem 1.1, as it allows one to move a vertex from $V_1$ to $V_2$ so that the new partition also satisfies (1). The bounds in (2) and (3) are (individually) tight; and the complete graphs $K_{2n+1}$ are the only extremal graphs (modulo isolated vertices) for Theorem 1.1. Bollobás and Scott [2] asked whether this can be extended to $k \geq 3$.

**Problem 1.2** *(Bollobás and Scott [2]) Does any graph $G$ of size $m$ have a partition of $V(G)$ into $V_1, \ldots, V_k$ that satisfies both* (1.1) *and* (1.2)*?*

A weaker version of Problem 1.2 is resolved in [8] by the authors, which we shall use in this paper to deal with small graphs.

**Theorem 1.3** *(Xu and Yu [8]) Let $G$ be a graph of size $m$, and let $k \geq 1$ be an integer. Then $V(G)$ admits a partition $V_1, \ldots, V_k$ such that*

(1) *for each $i \in \{1, \ldots k-1\}$ and for every $x \in V_i$, $|N(x) \cap (\bigcup_{j=i+1}^{k} V_j)| \geq (k-i)|N(x) \cap V_i|$,*

(2) *$e(V_i) \leq \frac{1}{k^2}m + \frac{k-1}{2k^2}h(m)$ for $i \in \{1, 2, \ldots, k\}$, and*

(3) *$e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{1}{2k}h(m)$.*

This result is stronger than the previous results when $k$ is restricted to powers of 2, and implies Theorem 1.1 when $k = 2$ . The main result of this paper is an affirmative answer to Problem 1.2.

**Theorem 1.4** *Let $G$ be a graph of size $m$, and let $k \geq 1$ be an integer. Then $V(G)$ admits a partition $V_1, \ldots, V_k$ such that*

(1) *for each $i \in \{1, \ldots k-1\}$ and for every $x \in V_i$, $|N(x) \cap (\bigcup_{j=i+1}^{k} V_j)| \geq (k-i)|N(x) \cap V_i|$,*

(2) *$e(V_i) \leq \frac{1}{k^2}m + \frac{k-1}{2k^2}h(m)$ for $i \in \{1, 2, \ldots, k\}$, and*

(3) *$e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{k-1}{2k}h(m) - \frac{17k}{8}$.*

The complete graphs $K_{kn+1}$ show that this is best possible, up to the $O(k)$ term. (We have not made an effort to optimize the $O(k)$ term.) Theorem 1.4 is closely related to another problem of Bollobás and Scott [2].

**Problem 1.5** *What is the largest $c(k)$ so that every graph $G$ with $m$ edges admits a partition of $V(G)$ into $V_1, \ldots, V_k$ such that for each $i \in \{1, \ldots, k\}$,*

$$\binom{k+1}{2} e(V_i) + c(k) \sum_{j \neq i} e(V_j) \leq m.$$

Bollobás and Scott note in [2] that $c(k) = k/2$ (if true) would be best possible. In Section 3 we shall see that for large complete graphs we must have $c(k) < k/2$. On the other hand, $c(k)$ can be arbitrarily close to $k/2$ for sufficiently large graphs. More precisely, we shall prove the following in Section 3.

**Corollary 1.6** *Let $G$ be a graph with $m \geq 1$ edges, and let $k \geq 1$ be an integer. Then there is a partition of $V(G)$ into $V_1, \ldots, V_k$ such that for $i \in \{1, \ldots, k\}$,*

$$\binom{k+1}{2} e(V_i) + \frac{k}{2} \sum_{j \neq i} e(V_j) \leq m + \frac{17k^2}{16}. \tag{1.3}$$

## 2 Proof of Theorem 1.4

The main idea of the proof is the same as that used in [8]. That is, we work with a partition $V_1, \ldots, V_k$ of $V(G)$ that satisfies (1) of Theorem 1.4 (called the **property** $\mathrm{P}(k)$ in [8]), and the property that $e(V_1) \geq e(V_i)$ for all $i \in \{1, \ldots, k\}$. Such a partition may be produced by maximizing $e(V_1, \ldots, V_k)$. If $e(V_1) \leq \frac{1}{k^2} m + \frac{k-1}{2k^2} h(m)$ then we have the desired partition. Otherwise, we move a vertex from $V_1$ to $\overline{V_1} := V(G) - V_1$, and (inductively) partition $G[\overline{V_1}]$ (the subgraph of $G$ induced by $\overline{V_1}$) into $k-1$ sets satisfying the property $\mathrm{P}(k-1)$.

We need the following simple observation, proved in [8].

**Lemma 2.1** *(Xu and Yu [8]) Let $G$ be a graph and $k \geq 2$ an integer, and let $V_1, \ldots, V_k$ be a partition of $V(G)$ such that for each $i \in \{1, \ldots k-1\}$ and for every $x \in V_i$, $|N(x) \cap (\bigcup_{j=i+1}^{k} V_j)| \geq (k-i)|N(x) \cap V_i|$. Then*

(1) *$e(V_1, \overline{V_1}) \geq 2(k-1)e(V_1)$, and*

(2) *for any $v \in V_1$, $e(V_1 - v, \overline{V_1} + v) \geq 2(k-1)e(V_1) - (k-2)|N(v) \cap V_1|$.*

In (2) above, $V_1 - v = V_1 \setminus \{v\}$ and $\overline{V_1} + v = \overline{V_1} \cup \{v\}$. We also need the following, which shows that Theorem 1.4 holds for small graphs.

**Lemma 2.2** *Theorem 1.4 holds when $m \leq 2k^2 - 1/8$.*

*Proof.* Note that $h(m) \leq 2k - 1/2$ when $m \leq 2k^2 - 1/8$. By Theorem 1.3, $V(G)$ admits a partition $V_1, \ldots, V_k$ such that (1) and (2) hold. In particular, for $1 \leq i \leq k$,

$$\begin{aligned} e(V_i) &\leq \frac{1}{k^2} m + \frac{k-1}{2k^2} h(m) \\ &\leq 2 - \frac{1}{8k^2} + \frac{k-1}{k} - \frac{k-1}{4k^2} \quad \text{(since } m \leq 2k^2 - 1/8\text{)} \\ &< 3. \end{aligned}$$

Therefore, we have $e(V_i) \leq 2$ for $i = 1, \ldots, k$. Hence, $e(V_1, \ldots, V_k) \geq m - 2k$, and so

$$e(V_1, \ldots, V_k) - \left( \frac{k-1}{k} m + \frac{k-1}{2k} h(m) - \frac{17k}{8} \right) \geq \frac{m}{k} - \frac{k-1}{2k} \sqrt{2m + 1/4} + \frac{k}{8} + \frac{k-1}{4k}.$$

Consider

$$g(m) := \frac{m}{k} - \frac{k-1}{2k} \sqrt{2m + 1/4} + \frac{k}{8} + \frac{k-1}{4k}$$

as a function of $m$ over the interval $[0, 2k^2 - 1/8]$. Differentiating with respect to $m$, we have

$$g'(m) = \frac{1}{k} - \frac{k-1}{2k\sqrt{2m+1/4}}.$$

So $g(m)$ has a unique critical point at $m = (k^2 - 2k)/8$ in $(0, 2k^2 - 1/8)$. Since $g''(m) > 0$ on $[0, 2k^2 - 1/8]$, we see that

$$g(m) \geq g((k^2-2k)/8) = \frac{k-2}{8} - \frac{k-1}{2k}\sqrt{\frac{k^2-2k+1}{4}} + \frac{k}{8} + \frac{k-1}{4k} = \frac{k-1}{2k} \geq 0.$$

Hence, (3) holds for $V_1, \ldots, V_k$. ∎

We now prove Theorem 1.4 by applying induction on $k$. For $k = 1$, Theorem 1.4 holds trivially, and for $k = 2$ it follows from Theorem 1.1. So we may assume $k \geq 3$.

As induction hypothesis, we assume that the assertion of Theorem 1.4 holds when partitioning any graph into $k-1$ sets. By Lemma 2.2, we may assume

$$m \geq 2k^2 - \frac{1}{8}. \tag{2.4}$$

Next we prove

*Claim 1.* $V(G)$ admits a partition $V_1, \ldots, V_k$ such that

- (1) and (3) of Theorem 1.4 hold for $V_1, \ldots, V_k$,
- $e(V_1) \geq \max_{2 \leq i \leq k}\{e(V_i)\}$, and
- subject to the above conditions, $e(V_1)$ is minimal.

To see this, let $V_1, \ldots, V_k$ be a partition of $V(G)$ maximizing $e(V_1, \ldots, V_k)$. Then by (1.1), the partition $V_1, \ldots, V_k$ satisfies (3) of Theorem 1.4. Moreover, for any $1 \leq i \neq j \leq k$ and for any $u \in V_i$, $|N(u) \cap V_i| \leq |N(u) \cap V_j|$. So the partition $V_1, \ldots, V_k$ satisfies (1) of Theorem 1.4. Without loss of generality, we may assume that

$$e(V_1) \geq \max_{2 \leq i \leq k}\{e(V_i)\}.$$

So we have Claim 1.

Let $e(V_1) = \frac{m}{k^2} + \alpha$. If $\alpha \leq \frac{k-1}{2k^2}h(m)$, we are done. So we may assume that

$$\alpha > \frac{k-1}{2k^2}h(m). \tag{2.5}$$

Let $H := G[V_1]$, $\delta$ the minimum nonzero degree in $H$, and $v$ a vertex of $H$ with degree $\delta$. Let $W_1 := V_1 - v$; so $\overline{W_1} = \overline{V_1} + v$ and $e(W_1) = e(V_1) - \delta = m/k^2 + \alpha - \delta$. Moreover, for every $u \in W_1$,

$$|N(u) \cap \overline{W_1}| \geq |N(u) \cap \overline{V_1}| \geq (k-1)|N(u) \cap V_1| \geq (k-1)|N(u) \cap W_1|. \tag{2.6}$$

For convenience, write $x := e(W_1, \overline{W_1})$ and $m' := e(\overline{W_1}) = m - e(W_1) - x$. By (2) of Lemma 2.1,

$$x \geq 2(k-1)e(V_1) - (k-2)\delta = 2(k-1)\left(\frac{m}{k^2} + \alpha\right) - (k-2)\delta. \tag{2.7}$$

Therefore, since $e(W_1) = e(V_1) - \delta = \frac{m}{k^2} + \alpha - \delta$, we have

$$m' \leq \frac{(k-1)^2}{k^2} m - (2k-1)\alpha + (k-1)\delta. \tag{2.8}$$

By induction hypothesis, $\overline{W_1}$ admits a partition $W_2, \ldots, W_k$ such that for every $i \in \{2, \ldots, k-1\}$ and for every $w \in W_i$,

$$\left| N(w) \cap \left( \bigcup_{j=i+1}^{k} W_j \right) \right| \geq (k-i)|N(w) \cap W_i|, \tag{2.9}$$

where the neighborhood is taken in $G[\overline{W_1}]$, and such that

$$e(W_i) \leq \frac{m'}{(k-1)^2} + \frac{k-2}{2(k-1)^2} h(m') \text{ for } i \in \{2, \ldots, k\}, \tag{2.10}$$

and

$$e(W_2, \ldots, W_k) \geq \frac{k-2}{k-1} m' + \frac{k-2}{2(k-1)} h(m') - \frac{17(k-1)}{8}. \tag{2.11}$$

We wish to show that $W_1, \ldots, W_k$ gives the desired partition of $V(G)$ for Theorem 1.4. By (2.6) and (2.9), we have

*Claim 2.* The partition $W_1, \ldots, W_k$ satisfies (1) of Theorem 1.4.

Next, we prove

*Claim 3.* $W_1, \ldots, W_k$ satisfies (3) of Theorem 1.4.
From (2.11), we see that

$$\begin{aligned} & e(W_1, \ldots, W_k) \\ = \; & e(W_1, \overline{W_1}) + e(W_2, \ldots, W_k) \\ \geq \; & x + \frac{k-2}{k-1} m' + \frac{k-2}{2(k-1)} h(m') - \frac{17(k-1)}{8}. \end{aligned}$$

We now prove that this lower bound on $e(W_1, \ldots, W_k)$ is at least $\frac{k-1}{k} m + \frac{k-1}{2k} h(m) - \frac{17k}{8}$. Let

$$f(x) := x + \frac{k-2}{k-1} m' + \frac{k-2}{2(k-1)} h(m') - \frac{17(k-1)}{8} - \frac{k-1}{k} m - \frac{k-1}{2k} h(m) + \frac{17k}{8}.$$

Recall that $m' = m - e(W_1) - x$ and $e(W_1) = m/k^2 + \alpha - \delta$. So $m' = m - x - m/k^2 - \alpha + \delta$, and $f(x)$ becomes

$$\frac{x}{k-1} - \frac{2m}{k^2} - \frac{k-2}{k-1}(\alpha - \delta) - \frac{k-1}{2k}\sqrt{2m + \frac{1}{4}} + \frac{k-2}{2(k-1)}\sqrt{\frac{2k^2-2}{k^2} m - 2x - 2\alpha + 2\delta + \frac{1}{4}} + \frac{17}{8} + \frac{1}{4k(k-1)}.$$

Since $x \le m - e(W_1) = \frac{k^2-1}{k^2}m - \alpha + \delta$ and because of (2.7), we have

$$a \le x \le b, \tag{2.12}$$

where $a := 2(k-1)\left(\frac{m}{k^2} + \alpha\right) - (k-2)\delta$ and $b := \frac{k^2-1}{k^2}m - \alpha + \delta$. Therefore, to prove Claim 3, it suffices to show that $f(x) \ge 0$ for $x \in [a,b]$.

Note that

$$f'(x) = \frac{1}{k-1} - \frac{k-2}{2(k-1)} \frac{1}{\sqrt{\frac{2k^2-2}{k^2}m - 2x - 2\alpha + 2\delta + \frac{1}{4}}}$$

and

$$f''(x) = -\frac{k-2}{2(k-1)}\left(\frac{2k^2-2}{k^2}m - 2x - 2\alpha + 2\delta + \frac{1}{4}\right)^{-\frac{3}{2}} < 0.$$

So $f'(x) = 0$ has at most one solution in $[a,b]$ at which (if exists) $f(x)$ reaches a local maximum. Thus, to prove $f(x) \ge 0$ for $x \in [a,b]$, it suffices to show $f(a) \ge 0$ and $f(b) \ge 0$.

First, we prove $f(b) \ge 0$. Suppose, to the contrary, $f(b) < 0$. Then substituting $x$ with $b = \frac{k^2-1}{k^2}m - \alpha + \delta$ in $f(x)$ and solving for $\alpha$ in $f(b) < 0$, we obtain

$$\alpha > \frac{k-1}{k^2}m + \delta - \frac{k-1}{2k}\sqrt{2m + \frac{1}{4}} + \frac{17}{8} + \frac{k-2}{4(k-1)} + \frac{1}{4k(k-1)}. \tag{2.13}$$

Recall that $e(W_1) = e(V_1) - \delta = \frac{m}{k^2} + \alpha - \delta$. So

$$\begin{aligned}
m &\ge x + e(W_1) \\
&\ge 2(k-1)\left(\frac{m}{k^2} + \alpha\right) - (k-2)\delta + \frac{m}{k^2} + \alpha - \delta \quad \text{(by (2.7))} \\
&= \frac{2k-1}{k^2}m + (2k-1)\alpha - (k-1)\delta \\
&> \frac{2k-1}{k^2}m - (k-1)\delta + (2k-1)\left(\frac{k-1}{k^2}m + \delta - \frac{k-1}{2k}\sqrt{2m + \frac{1}{4}}\right) \quad \text{(by (2.13))} \\
&> m + \frac{k-1}{k}\left(m - \frac{(2k-1)}{2}\sqrt{2m + \frac{1}{4}}\right) \\
&> m.
\end{aligned}$$

The final inequality holds because $m \ge 2k^2 - 1/8$ by (2.4), the function $l(m) := m - \frac{(2k-1)}{2}\sqrt{2m + \frac{1}{4}}$ is increasing when $m \ge 2k^2 - 1/8$, and $l(2k^2 - 1/8) > 0$. But this is a contradiction. So $f(b) \ge 0$.

Next we show $f(a) \geq 0$. Substituting $2(k-1)\left(\frac{m}{k^2}+\alpha\right)-(k-2)\delta$ for $a$ in $f(a)$, we get

$$\begin{aligned}
g(\alpha) &:= f(a) \\
&= \frac{1}{k-1}\left(\frac{2(k-1)}{k^2}m + 2(k-1)\alpha - (k-2)\delta\right) - \frac{2m}{k^2} - \frac{k-2}{k-1}(\alpha-\delta) \\
&\quad -\frac{k-1}{2k}\sqrt{2m+\frac{1}{4}} + \frac{17}{8} + \frac{1}{4k(k-1)} \\
&\quad +\frac{k-2}{2(k-1)}\sqrt{\frac{2(k^2-1)}{k^2}m - 2\alpha + 2\delta - \frac{4(k-1)}{k^2}m - 4(k-1)\alpha + 2(k-2)\delta + \frac{1}{4}} \\
&= \frac{k}{k-1}\alpha - \frac{k-1}{2k}\sqrt{2m+\frac{1}{4}} + \frac{17}{8} + \frac{1}{4k(k-1)} \\
&\quad +\frac{k-2}{2(k-1)}\sqrt{\frac{2(k-1)^2}{k^2}m - (4k-2)\alpha + 2(k-1)\delta + \frac{1}{4}}.
\end{aligned}$$

Note that $\alpha$ must satisfy $\frac{2(k-1)^2}{k^2}m - (4k-2)\alpha + 2(k-1)\delta + \frac{1}{4} \geq 0$, from which we deduce

$$\alpha \leq \beta := \frac{(k-1)^2}{k^2(2k-1)}m + \frac{k-1}{2k-1}\delta + \frac{1}{8(2k-1)}.$$

By (2.5),

$$\alpha > \theta := \frac{k-1}{2k^2}\left(\sqrt{2m+\frac{1}{4}} - \frac{1}{2}\right).$$

We now prove that $g(\alpha) \geq 0$ for $\alpha \in [\theta, \beta]$.

Note that $g(\beta) = \frac{k}{k-1}\beta - \frac{k-1}{2k}\sqrt{2m+1/4} + \frac{17}{8} + \frac{1}{4k(k-1)}$. Hence,

$$\begin{aligned}
g(\beta) &> \frac{k}{k-1}\left(\frac{(k-1)^2}{k^2(2k-1)}m + \frac{k-1}{2k-1}\delta + \frac{1}{8(2k-1)}\right) - \frac{k-1}{2k-1}\sqrt{2m+\frac{1}{4}} + \frac{17}{8} \\
&> \frac{k-1}{k(2k-1)}m - \frac{k-1}{2k-1}\sqrt{2m+\frac{1}{4}} + \frac{17}{8} \\
&= \frac{k-1}{2k-1}\left(\frac{m}{k} - \sqrt{2m+\frac{1}{4}}\right) + \frac{17}{8} \\
&> 0.
\end{aligned}$$

The last inequality follows from (2.4) that $m \geq 2k^2 - 1/8$, since the function $l(m) := \frac{m}{k} - \sqrt{2m+\frac{1}{4}}$ is an increasing function when $m \geq 2k^2 - 1/8$ and $l(2k^2-1/8) = -1/(8k)$.

On the other hand,

$$
\begin{aligned}
g(\theta) &= \frac{1}{2k}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right)-\frac{k-1}{2k}\sqrt{2m+\frac{1}{4}}+\frac{17}{8}+\frac{1}{4k(k-1)} \\
&\quad +\frac{k-2}{2(k-1)}\sqrt{\frac{2(k-1)^2m}{k^2}-\frac{(k-1)(2k-1)}{k^2}\left(\sqrt{2m+\frac{1}{4}}-\frac{1}{2}\right)+2(k-1)\delta+\frac{1}{4}} \\
&= -\frac{k-2}{2k}\sqrt{2m+\frac{1}{4}}-\frac{1}{4k}+\frac{17}{8}+\frac{1}{4k(k-1)} \\
&\quad +\frac{k-2}{2k}\sqrt{2m+\frac{k^2}{4(k-1)^2}-\frac{2k-1}{k-1}\sqrt{2m+\frac{1}{4}}+\frac{2k^2}{(k-1)}\delta+\frac{2k-1}{2(k-1)}} \\
&> \frac{17}{8}-\frac{1}{4k}+\frac{k-2}{2k}\left(\sqrt{2m+\frac{1}{4}-\frac{2k-1}{k-1}\sqrt{2m+\frac{1}{4}}}-\sqrt{2m+\frac{1}{4}}\right) \\
&= \frac{17}{8}-\frac{1}{4k}+\frac{k-2}{2k}\cdot\frac{-\frac{2k-1}{k-1}\sqrt{2m+\frac{1}{4}}}{\sqrt{2m+\frac{1}{4}-\frac{2k-1}{k-1}\sqrt{2m+\frac{1}{4}}}+\sqrt{2m+\frac{1}{4}}} \\
&> \frac{17}{8}-\frac{1}{4k}-\frac{k-2}{2k}\cdot\frac{2k-1}{k-1} \\
&> 0.
\end{aligned}
$$

Since

$$
g'(\alpha)=\frac{k}{k-1}-\frac{(k-2)(2k-1)}{2(k-1)}\left(\frac{2(k-1)^2}{k^2}m-(4k-2)\alpha+2(k-1)\delta+\frac{1}{4}\right)^{-\frac{1}{2}},
$$

$g'(\alpha)=0$ has at most one solution in $[\theta,\beta]$. Since

$$
g''(\alpha)=-\frac{(k-2)(2k-1)^2}{2(k-1)}\left(\frac{2(k-1)^2}{k^2}m-(4k-2)\alpha+2(k-1)\delta+\frac{1}{4}\right)^{-\frac{3}{2}}<0,
$$

$g(\theta)>0$ and $g(\beta)>0$ imply that $g(\alpha)>0$ for $\alpha\in[\theta,\beta]$. Therefore, $f(a)>0$. This together with $f(b)\geq 0$ shows that $f(x)\geq 0$ for $x\in[a,b]$, and hence the partition $W_1,W_2,\ldots,W_k$ satisfies (3) of Theorem 1.4, completing the proof of Claim 3.

If the partition $W_1,\ldots,W_k$ also satisfies (2) of Theorem 1.4, then by Claims 2 and 3, $W_1,\ldots,W_k$ is the desired partition of $V(G)$ for Theorem 1.4. So we may assume that

$$
\max_{1\leq i\leq k} e(W_i)>\frac{m}{k^2}+\frac{k-1}{2k^2}h(m). \tag{2.14}
$$

The rest of the proof of Theorem 1.4 is exactly the same as that in Section 3 of [8], starting from (3.9) in [8]. ∎

## 3 Corollary 1.6

Before we prove Corollary 1.6, we show that for large complete graphs one needs $c(k) < k/2$ for the constant in Problem 1.5. Note that for any partition $V_1, \ldots, V_k$ of a graph $G$, we have

$$\begin{aligned} \binom{k+1}{2} e(V_i) + \frac{k}{2} \sum_{j \neq i} e(V_j) &= \frac{k}{2} \left( (k+1)e(V_i) + \sum_{j \neq i} e(V_j) \right) \\ &= \frac{k}{2} \left( ke(V_i) + \sum_{j=1}^{k} e(V_j) \right) \\ &= \frac{k}{2} \left( ke(V_i) + m - e(V_1, ..., V_k) \right). \end{aligned}$$

Let $V_1, ..., V_k$ be a $k$-partition of $K_{kn+1}$ that maximizes $e(V_1, ..., V_k)$. Then we must have $|V_i| - |V_j| \in \{-1, 0, 1\}$. So we may choose notation so that $|V_1| = n+1$. Then

$$e(V_1) = \frac{1}{k^2} m + \frac{k-1}{2k^2} h(m) \text{ and } e(V_1, \ldots, V_k) = \frac{k-1}{k} m + \frac{k-1}{2k} h(m) - \frac{k^2 - 4k + 4}{8k}.$$

Now let $W_1, ..., W_k$ be an arbitrary $k$-partition of $K_{kn+1}$, and we may choose notation so that $e(W_1) = \max\{e(W_i)\}$. Then, $e(W_1) \geq e(V_1)$ and $e(W_1, ..., W_k) \leq e(V_1, ..., V_k)$. Hence,

$$\begin{aligned} &\binom{k+1}{2} e(W_1) + \frac{k}{2} \sum_{j=2}^{k} e(W_j) \\ = \; & \frac{k}{2} \left( ke(W_1) + m - e(W_1, ..., W_k) \right) \\ \geq \; & \frac{k}{2} \left( ke(V_1) + m - e(V_1, ..., V_k) \right) \\ = \; & m + \frac{(k-2)^2}{16}. \end{aligned}$$

Therefore, the term $c(k)$ in Problem 1.5 must satisfy $c(k) < k/2$. However, Corollary 1.6 shows that $\lim_{m \to \infty} c(k) = k/2$. Thus, in this sense, we have answered Problem 1.5.

*Proof of Corollary 1.6.* By Theorem 1.4, there is a partition $V_1, \ldots, V_k$ of $V(G)$ such that for each $i \in \{1, \ldots, k\}$,

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2} h(m),$$

and

$$e(V_1, \ldots, V_k) \geq \frac{k-1}{k} m + \frac{k-1}{2k} h(m) - \frac{17k}{8}.$$

Then

$$
\begin{aligned}
& \binom{k+1}{2} e(V_i) + \frac{k}{2}\sum_{j\neq i} e(V_j) \\
= \quad & \frac{k}{2}\left(ke(V_i) + m - e(V_1,\ldots,V_k)\right) \\
\leq \quad & \frac{k}{2}\left(k\left(\frac{m}{k^2} + \frac{k-1}{2k^2}h(m)\right) + m - \left(\frac{k-1}{k}m + \frac{k-1}{2k}h(m) - \frac{17k}{8}\right)\right) \\
= \quad & m + \frac{17k^2}{16}.
\end{aligned}
$$

∎

It is an interesting question to characterize the graphs $G$ such that for every $k$-partition $V_1,\ldots,V_k$ of $V(G)$ there exists some $i \in \{1,2,\ldots,k\}$ such that

$$\binom{k+1}{2} e(V_i) + \frac{k}{2}\sum_{j\neq i} e(V_j) > m.$$

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