Non-separating cycles avoiding specific vertices*

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Abstract

Thomassen proved that every $(k+3)$-connected graph G contains an induced cycle C such that $G - V(C)$ is k-connected, establishing a conjecture of Lovász. In general, one could ask the following question: For any positive integers k, l, does there exist a smallest positive integer $g(k,l)$ such that for any $g(k,l)$-connected graph G, any $X \subseteq V(G)$ with $|X| = k$, and any $e \in E(G - X)$, there is an induced cycle C in $G - X$ such that $e \in E(C)$ and $G - V(C)$ is l-connected? The case when $k = 0$ is a well-known conjecture of Lovász which is still open for $l \geq 3$. In this paper, we prove $g(k,1) \leq 10k + 1$ and $g(k,2) \leq 10k + 11$. We also consider a weaker version: For any positive integers k, l, is there a smallest positive integer $f(k,l)$ such that for every $f(k,l)$-connected graph G and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle C in $G - X$ such that $G - V(C)$ is l-connected? The case when $k = 0$ was studied by Thomassen. We prove $f(k,l) \leq 2k+l+2$ and $f(k,1) = k + 3$.

Keywords: connectivity, non-separating cycle, k-contractible edge

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1 Introduction

We begin with notation necessary for describing problems and results in this paper. Let G be a graph; we use $V(G)$ and $E(G)$ to denote its vertex and edge set, respectively. For any $e \in E(G)$, $V(e)$ denotes the set of vertices of G incident with e. By $H \subseteq G$ we mean that H is a subgraph of G, and we view any subset of vertices as a subgraph with no edges. For any $U \subseteq G$, the *neighborhood* of U, denoted by $N_G(U)$, is the set of vertices in $V(G) - V(U)$ adjacent to at least one vertex in U; and $N_G[U] := N_G(U) \cup U$ is the *closed neighborhood* of U in G. The *degree* $d_G(u)$ of a vertex u in G is $|N_G(\{u\})|$. If the graph G is clear from the context, the reference to G is usually omitted. For any $U \subseteq G$, $G[U]$ denotes the subgraph of G induced by $V(U)$ and we write $G - U := G[V(G) - V(U)]$. Let k be a positive integer. A graph G is k*-connected* if $|V(G)| \geq k+1$ and $G - U$ is connected for any $U \subseteq V(G)$ with $|U| < k$.

Lovász (see [14]) conjectured the existence of a function $g(k)$ such that for any positive integer k, any $g(k)$-connected graph G, and any distinct $s, t \in V(G)$, there exists a path P in G between s and t such that $G - V(P)$ is k-connected. This is equivalent to the problem for asking the existence of such a cycle C through a specified edge. A result of Tutte [16] shows that $g(1) = 3$; and that $g(2) = 5$ was proved independently by Chen, Gould and Yu [2] and by Kriesell [9]. Lovász's conjecture remains open for $k \geq 3$.

The result of Tutte [16] showing $g(1) = 3$ is actually stronger: For every 3-connected graph G, $e \in E(G)$, and $u \in G - V(e)$, there exists an induced cycle C in $G - u$ such that $e \in E(C)$ and $G - V(C)$ is connected. It is conjectured in [8] that there exists a function $f(k)$ such that every $f(k)$-connected graph G and every $s, t, u \in V(G)$, there is a path P between s and t in G and a k-connected subgraph H of G such that $u \in V(H)$ and $V(H) \cap V(P) = \emptyset$. It is further shown that this conjecture implies the above conjecture of Lovász. We feel that in potential applications one may need P and C to avoid certain vertices (see [17] for another example), and believe the following is true.

Conjecture 1.1. *For any positive integers k, l, there exists a smallest positive integer $g(k, l)$ such that for any $g(k, l)$-connected graph G, any $e \in E(G)$, and any $X \subseteq V(G) - V(e)$ with $|X| = k$, there exists an induced cycle C in $G - X$ such that $e \in E(C)$ and $G - V(C)$ is l-connected.*

When $k = 0$, Conjecture 1.1 is simply Lovász's conjecture mentioned above. We provide evidence to Conjecture 1.1 by proving

Theorem 1.2. *For any positive integer k, $g(k, 1) \leq 10k + 1$ and $g(k, 2) \leq 10k + 11$.*

The proof of Theorem 1.2 is given in Section 2, where we also mention a result about connectivity of k-linked graphs that is needed in our proof.

It turns out that asking the cycle in Lovász's conjecture to go through a specified edge is what makes the conjecture difficult. Thomassen [13] proved that for any positive integer k, every $(k + 3)$-connected graph contains a cycle C such that $G - V(C)$ is k-connected, establishing a conjecture of Lovász [10]. (This result was further strengthened by Egawa [3, 4] for graphs with girth at least 4 or 5.) We consider a similar relaxation of Conjecture 1.1: For any positive integers k, l, there exists a smallest positive integer $f(k, l)$ such that for any $f(k, l)$-connected graph G and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle C in

$G-X$ such that $G-V(C)$ is l-connected. In Section 3, we use the contractible edge technique to prove

Theorem 1.3. *For any positive integers k and l, $f(k,l) \leq 2k+l+2$.*

The result of Tutte mentioned above implies $f(1,1) \leq 3$. A cycle shows that $f(1,1) > 2$. Hence, $f(1,1) = 3$. The third result in our paper is the following; a stronger version is proved in Section 4.

Theorem 1.4. $f(k,1) = k+3$ *for* $k \geq 2$.

2 Proof of Theorem 1.2

To prove Theorem 1.2 we need a result about connectivity of k-linked graphs. A *linkage problem* in a graph G is a set of pairs of vertices of G, written as $\mathcal{L} = \{\{s_1,t_1\},\ldots,\{s_k,t_k\}\}$. A *solution* to $\mathcal{L}$ is a set of paths $\{P_1,\ldots,P_k\}$ such that s_i,t_i are the ends of P_i and, for any $i \neq j$ and any $x \in V(P_i) \cap V(P_j)$, $x \in \{s_i,t_i\} \cap \{s_j,t_j\}$. A graph G is called *k-linked* if every linkage problem with k pairwise disjoint pairs of vertices has a solution. A graph G is *strongly k-linked* if every linkage problem in G consisting of k pairs has a solution. Bollobás and Thomason [1] proved that every $22k$-connected graph is k-linked. In [15] Thomas and Wollan improve this bound to $10k$.

Lemma 2.1. (*Thomas and Wollan*). *Every $10k$-connected graph is k-linked.*

A result of Mader [11] implies that any k-linked graph on at least $2k$ vertices is strongly k-linked. Thus the following statement follows trivially from Lemma 2.1.

Corollary 2.2. *Every $10k$-connected graph is strongly k-linked.*

For a path P and $u,v \in V(P)$, we use uPv to denote the subpath of P between u and v, and we view P as a sequence of vertices. Let G be a graph and $B \subseteq G$. A *B-bridge* of G is a subgraph of G induced by all edges in a component of $G - V(B)$ and all edges from that component to B.

Proof of Theorem 1.2. We break this proof into two cases. In Case 1 we prove $g(k,1) \leq 10k+1$; and in Case 2, we show $g(k,2) \leq 10k+11$.

Case 1. Let G be a $(10k+1)$-connected graph, $e = st \in E(G)$, and $X = \{x_1,\ldots,x_k\} \subseteq G - V(e)$. We need to show that there is an induced cycle C in $G-X$ such that $e \in E(C)$ and $G - V(C)$ is connected.

Note that $G-e$ is $10k$-connected. Consider the linkage problem $\mathcal{L} = \big\{\{s,t\},\{x_1,x_2\},\ldots,\{x_{k-1},x_k\}\big\}$ in $G-e$, which has k pairs of vertices. By Corollary 2.2, there is a solution $\{P,Q_1,\ldots,Q_{k-1}\}$ to $\mathcal{L}$ such that P is from s to t, and Q_i is from x_i to x_{i+1} for $i = 1,\ldots,k-1$. We may assume all paths are induced in $G-e$. Note that $X \subseteq \bigcup_{i=1}^{k-1} Q_i$, and $\bigcup_{i=1}^{k-1} Q_i$ is a connected subgraph of $G - e - V(P)$.

Thus $G-e$ has an induced path P between s and t such that X is contained in a connected component C_0 of $G - e - V(P)$. Let $C_1, C_2, \ldots, C_q$ be the other components of $G - V(P)$ (if any) such that $|V(C_1)| \geq |V(C_2)| \geq |V(C_q)|$, and let $S(P) := (|V(C_0)|, |V(C_1)|, \ldots, |V(C_q)|)$. We choose P so that $S(P)$ is maximal with respect to the lexicographic ordering.

If $q = 0$ then $G - V(P)$ is connected; so $C := G[V(P)]$ is the desired cycle showing that $g(k, 1) \leq 10k+1$. We may thus assume that $q > 0$. Note that $|N(C_q) \cap V(P)| \geq 10k+1$. Choose two vertices $u, v \in N(C_q) \cap V(P)$ such that uPv is maximum. Without loss of generality we may assume that s, u, v, t occur on P in this order. Let Q be an induced path in $G[C_q \cup \{u, v\}]$ between u and v. Then $P' := sPuQvPt$ is an induced path in $G - X$ between s and t. Since G is $(10k + 1)$-connected, $uPv - \{u, v\}$ has a neighbor in $\bigcup_{i=0}^{q-1} V(C_i)$. So $S(P')$ is larger than $S(P)$, a contradiction.

Case 2. Let G be a $(10k + 11)$-connected graph, $e = st \in E(G)$, and $X = \{x_1, \ldots, x_k\} \subseteq V(G) - V(e)$. We show that $G - X$ contains an induced cycle C such that $e \in E(C)$ and $G - V(C)$ is 2-connected.

Note that $G-e$ is $10(k+1)$-connected. Consider the linkage problem $\mathcal{L} = \big\{\{s, t\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\}\big\}$, which has $k+1$ pairs of vertices. By Corollary 2.2, there is a solution $\{P, Q_1, \ldots, Q_k\}$ to $\mathcal{L}$ such that P is from s to t and Q_i is from x_i to x_{i+1} for $i = 1, \ldots, k$ (with $x_{k+1} = x_1$).

Since $X \subseteq \bigcup_{i=1}^{k} Q_i$ which is a cycle, X is contained in a 2-connected block B_0 of $G-V(P)$. Let $B_1, \ldots, B_q$ be the B_0-bridges of $G - V(P)$ (if any) such that $|V(B_1)| \geq |V(B_2)| \geq \ldots \geq |V(B_q)|$, and let $S(P) := (|V(B_0)|, |V(B_1)|, \ldots, |V(B_q)|)$. Note that $|V(B_i \cap B_0)| \leq 1$. We choose P so that $S(P)$ is maximal with respect to the lexicographic ordering. We may assume that P is induced in $G - e$.

If $q = 0$ then $G - V(P)$ is 2-connected; so $C := G[V(P)]$ is the desired cycle showing that $g(k, 2) \leq 10k + 11$. Hence we may assume $q > 0$. Since $|V(B_q \cap B_0)| \leq 1$ and $G - e$ is $10(k+1)$-connected, we may let $u, v \in N(B_q - V(B_q \cap B_0)) \cap V(P)$ such that uPv is maximal. Without loss of generality we may assume that s, u, v, t occur on P in order. Let Q be an induced path in $G[B_q \cup \{u, v\}] - V(B_q \cap B_0)$ between u and v. Then $P' := sPuQvPt$ is an induced path in $G - e - X$ between s and t. Since G is $(10k + 11)$-connected, $uPv - \{u, v\}$ has a neighbor in $\bigcup_{i=1}^{q-1} B_i$, or at least two neighbors in B_0. So $S(P')$ is larger than $S(P)$, a contradiction. ∎

The bounds in Theorem 1.2 are probably far from being best possible. One way to reduce these bounds is to find out the minimum connectivity of G that guarantees the existence of disjoint connected subgraphs P and H such that $\{s, t\} \subseteq V(P)$ and $X \subseteq V(H)$. When $|X| = 2$, G needs to be 6-connected by a result of Jung [6]; but this is not known for $|X| \geq 3$. The problem is more difficult if, in addition, one requires H to be l-connected for some $l \geq 2$.

3 Contractible edges

For our proof of Theorem 1.3, we need the concept of a contractible edge. Let G be a k-connected graph and $e \in E(G)$. We say that e is *k-contractible* if the graph obtained from G by contracting e, denoted by G/e, is k-connected.

Clearly every edge in a 1-connected graph (other than K_2) is 1-contractible. An edge e in a 2-connected graph G is 2-contractible iff $G - V(e)$ is connected; from this one can see that any 2-connected graph other than K_3 contains a lot of 2-contractible edges. Tutte [16] showed that K_4 is the only 3-connected graph which does not admit any 3-contractible edge. Fontet [5] and, independently, Martinov [12] proved that if a 4-connected graph contains no 4-contractible edge then it is the square of a cycle of length at least 5 or it is the line graph of a cyclically 4-edge-connected cubic graph. For general k, Thomassen [13] proved that if G is a k-connected graph with no triangles then G admits a k-contractible edge. This is then used in [13] to prove the following result (establishing a conjecture of Lovász [10]).

Lemma 3.1. (*Thomassen*). *For $k \geq 4$, every k-connected graph G contains an induced cycle C such that $G - V(C)$ is $(k-3)$-connected and $|N(u) \cap V(C)| \leq 3$ for all $u \in V(G) - V(C)$.*

For our proof of Theorem 1.3, we need to prove the following lemma about contractible edges avoiding a given set of vertices.

Lemma 3.2. *Let G be a k-connected graph, where $k \geq 4$, and let $X \subseteq V(G)$ such that $|X| \leq k/2 - 1$, $G - X$ has girth at least 5, and $|V(G) - N[X]| \geq |X| + 1$. Then there exists $u \in V(G) - N[X]$ such that u is incident with a k-contractible edge of G.*

Proof. Suppose that no vertex in $V(G) - N[X]$ is incident with a k-contractible edge of G. Let $u \in V(G) - N[X]$. Then for any $e \in E(G)$ incident with u, $V(e)$ is contained in a k-cut S_e of G. Let $\mathcal{Q}_u$ denote the collection of all quadruples (e, S_e, A_e, B_e) such that $e \in E(G)$ is incident with u, S_e is a k-cut of G with $V(e) \subseteq S_e$, A_e is a component of $G - S_e$, and $B_e := G - S_e - A_e$. We choose $e \in E(G)$ incident with u such that

(1) $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ and $|A_e|$ is minimal.

We claim that

(2) for each $u \in V(G) - N[X]$ and each $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ satisfying (1), $|A_e| \leq k - 2$.

For, suppose that there exist $u \in V(G) - N[X]$ and $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ satisfying (1) such that $|A_e| \geq k - 1$. Let $e = uv$. Since G is k-connected, u is adjacent to some $w \in V(A_e)$. Let $f = uw$. Then there exists a quadruple $(f, S_f, A_f, B_f) \in \mathcal{Q}_u$.

We claim that $A_e \cap A_f = \emptyset$ or $B_e \cap B_f = \emptyset$. For suppose $A_e \cap A_f \neq \emptyset$ and $B_e \cap B_f \neq \emptyset$. Then $T_1 := (S_e \cap V(A_f)) \cup (S_e \cap S_f) \cup (V(A_e) \cap S_f)$ is a cut of G and contains $V(f)$, and $T_2 := (S_e \cap V(B_f)) \cup (S_e \cap S_f) \cup (V(B_e) \cap S_f)$ is a cut of G. Thus $|T_1| \geq k + 1$ (by (1)) and $|T_2| \geq k$ (since G is k-connected). So

$$2k = |S_e| + |S_f| = |T_1| + |T_2| \geq 2k + 1,$$

a contradiction.

Similarly, we can show that $A_e \cap B_f = \emptyset$ or $A_f \cap B_e = \emptyset$.

Suppose $B_e \cap B_f = \emptyset$. If $A_f \cap B_e = \emptyset$ then by (1), $k - 1 \leq |A_e| \leq |B_e| = |V(B_e) \cap S_f| \leq k - 2$, a contradiction. So $A_f \cap B_e \neq \emptyset$. Thus $A_e \cap B_f = \emptyset$, and $S_e \cap V(A_f) \neq \emptyset$ as G is k-connected. Hence, by (1), $k - 1 \leq |A_e| \leq |B_f| = |V(B_f) \cap S_e| \leq k - 2$, a contradiction.

Therefore, $B_e \cap B_f \neq \emptyset$. Similarly, $B_e \cap A_f \neq \emptyset$. So $A_e \cap A_f = \emptyset$ and $A_e \cap B_f = \emptyset$. Moreover, $S_e \cap V(A_f) \neq \emptyset$ as G is k-connected; so $S_f \cap V(B_e) \neq \emptyset$ as G is k-connected. Hence $|A_e| \leq k - 2$, a contradiction which completes the proof of (2).

Since G is k-connected, u has a neighbor in A_e; hence, since $u \in V(G) - N[X]$, $A_e - X \neq \emptyset$. In fact,

(3) for any $u \in V(G) - N[X]$ and for any $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ satisfying (1), we have $|V(A_e) - X| = 1$.

First, suppose $|V(A_e) - X| \geq 3$, and let $w_1, w_2, w_3 \in V(A_e) - X$. We wish to estimate $|N(w_1) \cup N(w_2) \cup N(w_3)|$. Applying the principle of inclusion and exclusion, we have

$$|N(w_1) \cup N(w_2) \cup N(w_3)| = \left(\sum_{i=1}^{3} |N(w_i)|\right) - \left(\sum_{1\leq i<j\leq 3} |N(w_i) \cap N(w_j)|\right) + |N(w_1) \cap N(w_2) \cap N(w_3)|.$$

Since $G-X$ has girth at least 5, $|N(w_i) \cap N(w_j) - X| \leq 1$, and if $|N(w_1) \cap N(w_2) \cap N(w_3) - X| \neq 0$ then $N(w_1) \cap N(w_2) - X = N(w_2) \cap N(w_3) - X = N(w_1) \cap N(w_3) - X$. So

$$\left|\bigcup_{1\leq i<j\leq 3} (N(w_i) \cap N(w_j) - X)\right| + |N(w_1) \cap N(w_2) \cap N(w_3) - X| \leq 3.$$

Note that

$$\left|\bigcup_{1\leq i<j\leq 3} N(w_i) \cap N(w_j) \cap X\right| + |N(w_1) \cap N(w_2) \cap N(w_3) \cap X| \leq 2|X|.$$

Hence,

$$\begin{aligned} & \left(\sum_{1\leq i<j\leq 3} |N(w_i) \cap N(w_j)|\right) - |N(w_1) \cap N(w_2) \cap N(w_3)| \\ = & \left|\bigcup_{1\leq i<j\leq 3} (N(w_i) \cap N(w_j))\right| + |N(w_1) \cap N(w_2) \cap N(w_3)| \\ \leq & \ 2|X| + 3. \end{aligned}$$

To see the equality above, we note that each vertex in exactly two of $N(w_1), N(w_2), N(w_3)$ is counted exactly once on both sides, and each vertex belonging to all three is counted exactly twice on both sides. Therefore, since $|X| \leq k/2 - 1$, we have

$$|N(w_1) \cup N(w_2) \cup N(w_3)| \geq \sum_{i=1}^{3} |N(w_i)| - (2|X| + 3) \geq 3k - (2|X| + 3) \geq 2k - 1.$$

Since $|S_e| = k$ and $N(w_i) \subseteq V(A_e) \cup S_e$ for $i = 1, 2, 3$, we have $|A_e| \geq k - 1$, contradicting (2).

Now suppose $|V(A_e) - X| = 2$, and let $V(A_e) - X = \{w_1, w_2\}$. If $w_1w_2 \notin E(G)$ then, since G is k-connected, each w_i has at least $k - |X|$ neighbors in $S_e - X$; so w_1 and w_2 have at least $2(k - |X|) - k \geq 2$ common neighbors in $S_e - X$ (since $|X| \leq k/2 - 1$), which forces a cycle in $G - X$ of length at most 4, a contradiction. So $w_1w_2 \in E(G)$; and thus $N(w_1) \cap N(w_2) \cap (S_e - X) = \emptyset$. Hence,

$$\begin{aligned} k & \geq |S_e - X| \\ & \geq |(N(w_1) \cup N(w_2)) \cap (S_e - X)| \\ & = |N(w_1) \cap (S_e - X)| + |N(w_2) \cap (S_e - X)| \\ & \geq 2(k - |X| - 1) \\ & \geq k \quad \text{(since } |X| \leq k/2 - 1). \end{aligned}$$

This inequality shows that $\big(N(w_1) \cup N(w_2)\big) \cap (S_e - X) = S_e - X = S_e$. Since $u, v \in S_e - X$, $G[\{u, v, w_1, w_2\}]$ contains a cycle, a contradiction, which completes the proof of (3).

So for any $u \in V(G) - N[X]$ and for any $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ satisfying (1), we let $V(A_e) - X = \{w_u\}$. Let $X_u = V(A_e) \cap X$. Since $u \in V(G) - N[X]$, we have

(4) $uw_u \in E(G)$, $N(X_u) = (S_e - \{u\}) \cup \{w_u\}$, and $N(w_u) - N[X] = \{u\}$.

We claim that

(5) for any distinct $u, u' \in V(G) - N[X]$, any $(e, S_e, A_e, B_e) \in \mathcal{Q}_u$ satisfying (1), and any $(f, S_f, A_f, B_f) \in \mathcal{Q}_{u'}$ satisfying (1), we have $X_u \cap X_{u'} = \emptyset$, where $X_u = X \cap V(A_e)$ and $X_{u'} = X \cap V(A_f)$.

Suppose $X_u \cap X_{u'} \neq \emptyset$. If $w_u = w_{u'}$ then, by (4), $\{u\} = N(w_u) - N[X] = N(w_{u'}) - N[X] = \{u'\}$, a contradiction. So $w_u \neq w_{u'}$.

Since $u, u' \notin N[X]$, $N[X_u \cup X_{u'}] \neq V(G)$. Therefore, since G is k-connected, $|N(X_u \cup X_{u'})| \geq k$ and, by (4),

$$2k \leq |N(X_u \cup X_{u'})| + |N(X_u \cap X_{u'})| \leq |N(X_u)| + |N(X_{u'})| = 2k.$$

It follows that $|N(X_u \cup X_{u'})| = k$. Let $F_u := N(w_u) - (X \cup \{u\})$ and $F_{u'} := N(w_{u'}) - (X \cup \{u'\})$. Then $F_u \subseteq N(X_u) - X$ (by (4)) and $|F_u| \geq k - |X| - 1$, and $F_{u'} \subseteq N(X_{u'}) - X$ (by (4)) and $|F_{u'}| \geq k - |X| - 1$. Note that $w_u \neq u'$ and $w_{u'} \neq u$, since $u, u' \notin N[X]$.

If $w_u w_{u'} \notin E(G)$ then $w_u, w_{u'} \notin F_u \cup F_{u'}$. Hence, since $w_u \in N(X_u) - X$ and $w_{u'} \in N(X_{u'}) - X$, $|F_u \cup F_{u'}| \leq |N(X_u \cup X_{u'}) - X| - 2 \leq k - 2$ (as $|N(X_u \cup X_{u'})| = k$). So $|F_u \cap F_{u'}| = |F_u| + |F_{u'}| - |F_u \cup F_{u'}| \geq 2(k - |X| - 1) - (k - 2) \geq 2$ (as $|X| \leq k/2 - 1$), which forces a cycle of length 4 in $G - X$, a contradiction.

So $w_u w_{u'} \in E(G)$. Then $F_u \cap F_{u'} = \emptyset$, since the girth of $G - X$ is at least 5. Thus $|F_u \cup F_{u'}| = |F_u| + |F_{u'}| \geq 2(k - |X| - 1) \geq k$ (as $|X| \leq k/2 - 1$). On the other hand, $|F_u \cup F_{u'}| \leq |N(X_u \cup X_{u'})| = k$. So $F_u \cup F_{u'} = N(X_u \cup X_{u'})$. Let $e = uv$. Then by (4), $v \in N(X_u) - X \subseteq N(X_u \cup X_{u'})$. Hence $v \in F_{u'}$ as $v \notin F_u$ (since $w_u v \notin E(G)$). Now $uw_u w_{u'} vu$ is a cycle in $G - X$, a contradiction, which completes the proof of (5).

Since $X_u \neq \emptyset$ and because of (5), $|V(G) - N[X]| \leq |\bigcup_{u \in V(G) - N[X]} X_u| \leq |X|$, contradicting the assumption that $|V(G) - N[X]| \geq |X| + 1$. ∎

When $|X| = 1$, we have a stronger version of Lemma 3.2.

Lemma 3.3. *Let G be a k-connected graph, where $k \geq 4$, and let $x \in V(G)$ such that $G - x$ has girth at least 5. Then there is a k-contractible edge not incident with x.*

Proof. If $|V(G) - N[x]| \geq 2$ then, by Lemma 3.2, there is a k-contractible edge in G not incident with X. So we may assume $|V(G) - N[x]| \leq 1$. Then $d_G(x) \geq |V(G - x)| - 1$. Let $u, v, w \in V(G - x)$ be distinct such that $uv, vw \in E(G - x)$. Since $G - x$ has girth at least 5 and $G - x$ is at least $(k-1)$-connected, $|V(G - x)| \geq (d_{G-x}(u) - 1) + (d_{G-x}(v) - 2) + (d_{G-x}(w) - 1) + 3 \geq 3k - 4$. Since $k \geq 4$,

(1) $d_G(x) \geq |V(G - x)| - 1 \geq 3k - 5 \geq k + 3$.

Suppose that all k-contractible edges in G are incident with x. Then for any edge e not incident with x, there is a k-cut S_e containing $V(e)$. Let $\mathcal{Q}$ denote the set of all quadruples (e, S_e, A_e, B_e) such that $e \in E(G)$ is not incident with x, S_e is a k-cut of G containing $V(e)$, A_e is a component of $G - S_e$, and $B_e = G - S_e - V(A_e)$. We may choose an edge e not incident with x and choose $(e, S_e, A_e, B_e) \in \mathcal{Q}$, such that

(2) $|A_e|$ is minimal.

By (1) and by our assumption that $G - x$ has girth at least 5, we see that $|A_e| \geq 2$. Furthermore,

(3) $|A_e| \geq k - 1$.

Since $|A_e| \geq 2$, $A_e - \{x\} \neq \emptyset$. If $|A_e - \{x\}| = 1$, then $x \in A_e$ and thus $d_G(x) \leq |S_e \cup A_e - \{x\}| = k + 1$, contradicting (1). Hence $|A_e - \{x\}| \geq 2$. Let $w_1, w_2 \in V(A_e) - \{x\}$. Since $G - x$ has no cycle of length less than 5, $|(N(w_1) \cap N(w_2)) - \{x\}| \leq 1$ if $w_1 w_2 \notin E(G)$ and $|(N(w_1) \cap N(w_2)) - \{x\}| = 0$ if $w_1 w_2 \in E(G)$. So $|N[w_1] \cup N[w_2]| \geq 2k - 1$. Hence $|A_e| \geq k - 1$, since $|S_e| = k$ and $N[w_1] \cup N[w_2] \subseteq V(A_e) \cup S_e$, which completes the proof of (3).

Since $|A_e| \geq k - 1 \geq 3$, it is easy to see that there is an edge f in $G - x$ such that $V(f) \cap V(A_e) \neq \emptyset$ and $V(f) \cap S_e \neq \emptyset$. Then the same proof for (2) in the proof of Lemma 3.2 works here and gives a contradiction. ∎

We now can prove the following result from which Theorem 1.3 follows directly. Let $\mathcal{F}$ be a family of graphs. A graph G is said to be $\mathcal{F}$-free if no induced subgraph of G is isomorphic to a graph in $\mathcal{F}$.

Theorem 3.4. *For any positive integers k, l, any $(2k + l + 2)$-connected graph G, and any $X \subseteq V(G)$ with $|X| = k$, there is an induced cycle C in $G - X$ such that $G - V(C)$ is l-connected. Moreover, if $G - X$ is $\{K_3, K_{2,3}\}$-free then $|N(u) \cap V(C)| \leq 1$ for any $u \notin V(C) \cup X$.*

Proof. Let G be a $(2k + l + 2)$-connected graph and let $X \subseteq V(G)$ with $|X| = k$. We proceed by induction on $|V(G)|$. If $|V(G)| = 2k + l + 3$, then G is complete; so any triangle in $G - X$ gives the desired cycle. We may thus assume that $|V(G)| \geq 2k + l + 4$ and the assertion holds for graphs of order less than $|V(G)|$. We may also assume that

(1) the girth of $G - X$ is at least 5.

For, suppose the girth of $G - X$ is at most 4, and let C be a cycle in $G - X$ with $|C|$ minimum. Then $|C| \leq 4$, C is induced and $G - V(C)$ has connectivity at least $2k + l + 2 - 4 = 2(k-1) + l \geq l$, as $k \geq 1$. Now assume that $G - X$ is $\{K_3, K_{2,3}\}$-free. Then $|C| = 4$, and $|N(u) \cap V(C)| \leq 1$ for each $u \notin V(C) \cup X$.

Next, we show that

(2) when $k \geq 2$, we may assume that $|V(G) - N[X]| \geq |X| + 1 = k + 1$.

Suppose $k \geq 2$ and $|V(G) - N[X]| \leq k$. For each $x \in X$, $G - x$ has connectivity at least $2k + l + 2 - 1 > 2(k-1) + l + 2$; so by induction there is an induced cycle C_x in $G - X$ such that $(G - x) - V(C_x)$ is l-connected and $|N(u) \cap V(C_x)| \leq 1$ for any $u \notin V(C_x) \cup X$. Hence

$$|V(G - X) - V(C_x)| \geq \sum_{u \in V(C_x)} (d_G(u) - k - 2) \geq (k + l)|V(C_x)|.$$

It follows that $|V(C_x)| \leq |V(G - X)|/(k + l + 1)$. If $|N(x) \cap V((G - x) - V(C_x))| \geq l$ then $G - V(C_x)$ is l-connected. So we may assume that for any $x \in X$,

$$\begin{aligned} l &> |N(x) \cap V((G - x) - V(C_x))| \\ &= d_G(x) - |N(x) \cap V(C_x)| \\ &\geq d_G(x) - |V(C_x)| \\ &\geq d_G(x) - |V(G - X)|/(k + l + 1). \end{aligned}$$

Hence,

$$|V(G - X)| > (d_G(x) - l)(k + l + 1).$$

So

$$k|V(G - X)| \geq \left(\left(\sum_{x \in X} d_G(x) \right) - kl \right) (k + l + 1).$$

Since we assume $|V(G) - N[X]| \leq k$, $\sum_{x \in X} d_G(x) \geq |N(X)| \geq |V(G - X)| - k$. Thus $k|V(G - X)| \geq (|V(G - X)| - k - kl)(k + l + 1)$, and hence $|V(G - X)| \leq k(k + l + 1)$. Therefore,

$$(2k + 2)(k + l + 1) \leq (d_G(x) - l)(k + l + 1) < |V(G - X)| \leq k(k + l + 1),$$

a contradiction which completes the proof of (2).

By (2), we may apply Lemma 3.2 or Lemma 3.3 to conclude that there is a $(2k + l + 2)$-contractible edge $e = uv \in E(G - X)$. Thus G/e is $(2k + l + 2)$-connected and $X \subseteq V(G/e)$. Let z denote the vertex of G/e resulted from the contraction of e. By (1), G/e is also $\{K_3, K_{2,3}\}$-free. So by induction, $G/e - X$ contains an induced cycle C' such that $G/e - V(C')$ is l-connected and $|N(u) \cap V(C')| \leq 1$ for all $u \in V(G/e) - (X \cup V(C'))$.

We may view the edges of C' as edges of G, and let C be an induced cycle in the subgraph of G induced by $E(C') \cup \{e\}$. So $C \subseteq G - X$.

Let $v \in V(G) - (X \cup V(C))$. We claim that $|N_G(v) \cap V(C)| \leq 1$. If $v \notin V(e)$ then $|N_G(v) \cap V(e)| \leq 1$ (by (1)); so $|N_G(v) \cap V(C)| \leq |N_{G/e}(v) \cap V(C')| \leq 1$. Now assume $v \in V(e)$. If $z \notin V(C')$ then $|N_G(v) \cap V(C)| \leq |N_{G/e}(z) \cap V(C')| \leq 1$. So assume $z \in V(C')$. Then, since C' is induced in G/e, it follows from (1) that $|N_G(v) \cap V(C)| \leq 1$.

It remains to show that $G - V(C)$ is l-connected. For, suppose S is cut in $G - V(C)$ with $|S| \leq l - 1$. Then $|S \cap V(e)| = 1$ as otherwise S or $(S - V(e)) \cup \{z\}$ would be a cut in $G/e - V(C')$ (which is l-connected). For the same reason, $G - S$ has exactly two components, one of which consists of the vertex in $V(e) - S$, say v. But v has degree at most $1 + (l - 1) = l < 2k + l + 2$, a contradiction. ∎

4 Determining $f(k, 1)$

In this section, we prove Theorem 1.4. Actually we prove a result stronger than Theorem 1.4, and our approach is to find a connected subgraph T of G such that $X \subseteq V(T)$, and then find a cycle in $G - T$. This leads to the following concept.

Let G be a graph and $X \subseteq V(G)$. We define an *X-tree* in G as a minimal connected induced subgraph of G containing X. When $|X| = 1$, there is a unique X-tree, namely $G[X]$. When $|X| = 2$, an X-tree is simply an induced path in G between the vertices in X. However, for $|X| \geq 3$, an X-tree need not be a tree.

Let T be an X-tree in G. Then by minimality of T, any vertex in $V(T) - X$ is a cut vertex of T and, in particular, every vertex of degree at most 1 in T must be in X. Given any subgraph H of $G - V(T)$, we define a partition of $V(T)$ as follows. Recall the definition of a bridge in Section 2 (following Corollary 2.2). For each $u \in V(T)$, let C_u denote the maximal union of u-bridges of T such that $N(C_u - u) \cap V(H) = \emptyset$ and let $C_u = \emptyset$ if no such u-bridge of T exists. (Thus, by definition, $C_u \subseteq T$; so $C_u \cap H = \emptyset$.) We say that u is *H-maximal* if for any $v \in V(T) - \{u\}$, C_u is not contained in C_v. Define

$$V_1 = \bigcup V(C_u - u), \text{ where the union is taken over all } H\text{-maximal } u;$$
$$V_2 = (X - V_1) \cup \{u : u \text{ is } H\text{-maximal}\};$$
$$V_3 = V(T) - (V_1 \cup V_2).$$

We say that V_1, V_2, V_3 is the *H-partition* of $V(T)$. The following lemma summarizes a few properties about H-partitions and X-trees. In particular, property (1) implies that $V_1 \cap V_2 = \emptyset$; so V_1, V_2, V_3 is a partition of $V(T)$ (here we allow sets in a partition to be empty).

Lemma 4.1. *Let G be a connected graph, $X \subseteq V(G)$, T an X-tree in G, $H \subseteq G - V(T)$, and V_1, V_2, V_3 the H-partition of $V(T)$. Then*

(1) *if $u, v \in V_2$ are distinct and H-maximal then $C_u \cap C_v = \emptyset$,*

(2) *$N(V_1) \cap (V(H) \cup V_3) = \emptyset$,*

(3) *$X \subseteq V_1 \cup V_2$,*

(4) *for any $u \in V_3$, each component of $T - u$ contains a neighbor of H (so if H is connected then $G[T \cup H] - u$ is connected),*

(5) *$|V_2| \leq |X|$, and if $|V_2| = |X|$ then every component of $G[V_1 \cup V_2] - E(G[V_2])$ is a path between X and V_2 with internal vertices (if any) in V_1.*

Proof. Let $u, v \in V(T)$ be distinct and H-maximal. If $u \notin V(C_v)$ and $v \notin V(C_u)$ then u belongs to a v-bridge of T not contained in C_v, and v belongs to a u-bridge of T not contained in C_u; in this case it is easy to see that $C_u \cap C_v = \emptyset$. So by symmetry, we may assume $u \in V(C_v)$. By the maximality of C_v, any v-bridge of T not contained in C_v has a neighbor in H, which shows that $C_u \subseteq C_v$, contradicting the H-maximality of u. So we have (1).

We have (2) and (3) from the definition of the H-partition of $V(T)$. Now let $u \in V_3$. Then by definition, every component of $T - u$ contains a neighbor of H; so if H is connected then $G[T \cup H] - u$ is connected, and we have (4).

If $V_2 \subseteq X$ then $|V_2| \leq |X|$. Now assume $V_2 \not\subseteq X$, and let $u_1, \ldots, u_t \in V_2 - X$. Then by definition, each u_i is H-maximal. Thus, by (1), $C_{u_i} \cap C_{u_j} = \emptyset$ whenever $i \neq j$. Moreover, since T is an X-tree, every component of $C_{u_i} - u_i$ intersects X. Therefore, $|V_2| \leq |X|$. Next, assume $|V_2| = |X|$. If $V_2 = X$ then by definition $V_1 = \emptyset$; so (5) holds (by viewing each each vertex in $X = V_2$ as a trivial path between X and V_2. Hence, let $u_1, \ldots, u_t \in V_2 - X$. Then each $C_{u_i} - u_i$ is connected and contains only one vertex from X; hence by the minimality of T, C_{u_i} must be a path from $u_i \in V_2$ to some vertex in X and with all internal vertices in V_1. Thus we have (5), viewing each vertex in $X \cap V_2$ as a trivial path between X and V_2. ∎

In order to state our next result, we need the concept of a *k-fold wheel with center X*, which is defined as a graph obtained from the disjoint union of a cycle C and a disconnected graph X with $|V(X)| = k$ by adding all possible edges between $V(C)$ and $V(X)$; and $V(X)$ (or X) is called the *center* of the wheel. A 2-fold wheel is also called a *double* wheel. Note that a k-fold wheel with center X is not $(|X| + 3)$-connected, but if the cycle $G - X$ has at least $|X| + 2$ vertices then it is $(|X| + 2)$-connected.

Kawarabayashi, Lee and Yu [7] proved that if G is a 4-connected graph and $u, v \in V(G)$ are distinct, then either G is a 2-fold wheel with center $\{u, v\}$, or G has a path P between u and v such that $G - V(P)$ is 2-connected. This result together with a result in [2, 9] implies $f(2, 1) = 5$.

Recall the result of Tutte that if G is a 3-connected graph and $x \in V(G)$ then there is a cycle C in $G - x$ such that $G - V(C)$ is connected. From this we can deduce $f(1, 1) = 3$. Therefore, to prove Theorem 1.4 it suffices to consider $k \geq 3$.

Lemma 4.2. *Let G be a $(k+2)$-connected graph and $X \subseteq V(G)$ with $|X| = k \geq 3$. Suppose G is not a k-fold wheel with center $G[X]$. Then there exists an induced cycle C in G such that X is contained in a component of $G - V(C)$.*

Proof. For an X-tree T in G, let $D_1, \ldots, D_r$ be the components of $G - V(T)$ such that $|V(D_1)| \geq |V(D_2)| \geq \ldots \geq |V(D_r)|$. We choose T such that

(1) $|V(T)|$ is minimum, and

(2) subject to (1), $S(T) := (|V(D_1)|, |V(D_2)|, \ldots, |V(D_r)|)$ is maximal with respect to the lexicographic ordering.

Suppose that the assertion of Lemma 4.2 is false for G, X. Then each D_i is a tree. Let $x \in V(D_r)$ with degree at most 1 in D_r, and let V_1, V_2, V_3 be the x-partition of $V(T)$. (So Lemma 4.1 holds for V_1, V_2, V_3.) Then $N(x) \cap V(T) \subseteq V_2 \cup V_3$. (In this proof N without subscript is used to denote the neighborhood in G.) Since $d_G(x) \geq k + 2$ and $|V_2| \leq k$ (by Lemma 4.1(5)), we see that $N(x) \cap V_3 \neq \emptyset$; in particular, $V_3 \neq \emptyset$. We claim that

(3) each $u \in V_3$ has at most one neighbor in $V(D_r) - \{x\}$ and no neighbor in $V(D_i)$ for $i = 1, \ldots, r - 1$.

Since $G[V(T) \cup \{x\}] - u$ is connected (by Lemma 4.1(4)), $G[V(T) \cup \{x\}] - u$ contains an X-tree, say T'. If u has two neighbors in $V(D_r) - \{x\}$, say y_1, y_2, then the edges uy_1, uy_2 and a path in $D_r - x$ between y_1 and y_2 form a cycle disjoint from T', a contradiction to our assumption that Lemma 4.2 fails with G, X. If u has a neighbor in D_i for some $i \le r-1$, then $S(T')$ is larger than $S(T)$, a contradiction to (1) or (2). So we have (3).

We will show $|V_3| = 1$. Let δ denote the minimum degree of $G[V_3]$, and let $u \in V_3$ have degree δ in $G[V_3]$. By (3), u has at most 2 neighbors outside T (u may be adjacent to x and a vertex in $D_r - x$); so by Lemma 4.1(2), $|N(u) \cap V_2| \ge (k+2) - \delta - 2 = k - \delta$. Let $A := N(u) \cap V_2$ and $B := V_2 - A$. Thus

(4) $|A| \ge k - \delta$ and $|B| \le |V_2| - (k - \delta) \le \delta$.

(5) For any edge wz of $G[V_3] - u$, $\{w, z\}$ is a 2-cut of T, and $T - \{w, z\}$ has a component F_{wz} such that $V(F_{wz}) \cap V_3 = \emptyset$, $N_T(F_{wz}) = \{w, z\}$, $V(F_{wz}) \cap A = \emptyset$, and $V(F_{wz}) \cap B \ne \emptyset$.

Since $X \subseteq V_1 \cup V_2$ (by Lemma 4.1(3)), if follows from (1) that $G[V(T) \cup \{x\}] - \{w, z\}$ is not connected and hence has a component F_{wz} disjoint from $A \cup \{u, x\}$. So $V(F_{wz}) \cap A = \emptyset$ and F_{wz} is a component of $T - \{w, z\}$. Moreover, $w \in N_T(F_{wz})$ as, otherwise, $G[V(T) \cup \{x\}] - z$ is not connected, contradicting Lemma 4.1(4) (since $z \in V_3$). Similarly, $z \in N_T(F_{wz})$. So we have $N_T(F_{wz}) = \{w, z\}$.

If there exists $z' \in V(F_{wz}) \cap V_3$ then, since $z' \notin X$, $T - z'$ has a component, say F, which is inside F_{wz}. Now F is also a component of $G[V(T) \cup \{x\}] - z'$, contradicting Lemma 4.1(4). Hence, $V(F_{wz}) \cap V_3 = \emptyset$.

Therefore, since $V(F_{wz}) \cap A = \emptyset$, $V(F_{wz}) \cap B \ne \emptyset$ (by Lemma 4.1(2)), completing the proof of (5).

(6) If $|V_3| \ge 2$ then there exist $v \in V_3 - \{u\}$ and a component F_v of $T - v$ such that $V(F_v) \cap V_3 = \emptyset$, $N_T(F_v) = \{v\}$, $V(F_v) \cap A = \emptyset$, and $V(F_v) \cap B \ne \emptyset$.

Suppose $|V_3| \ge 2$. Then $T - u$ has a component, say F, containing at least one vertex in V_3. Now, for any $v \in V(F) \cap V_3$, at least one component of $T - v$, say F_v, is contained in F. Choose v and F_v so that $|V(F_v)|$ is minimum. Then $V(F_v) \cap V_3 = \emptyset$, and $N_T(F_v) = \{v\}$. So $u \notin F_v$, which implies $V(F_v) \cap A = \emptyset$. Hence, $V(F_v) \cap B \ne \emptyset$ by Lemma 4.1(2), completing the proof of (6).

(7) $|V_2| = |X|$, and $G[V_3] \cong K_s$ for some $s \in \{1, 2\}$.

Let $|V_3| = t$ and $|E(G[V_3])| = m$. Suppose $t = 1$. Then $G[V_3] \cong K_1$ and $V_3 = \{u\}$. So by Lemma 4.1(2) and by (3), $|V_2| \ge d_T(u) \ge d_G(u) - 2 \ge (k+2) - 2 = k$. It follows from Lemma 4.1(5) that $|V_2| = k$, and we have (7).

Thus we may assume $t \ge 2$. Let $e, f \in \mathcal{S} := \{wz : wz \text{ is an edge of } G[V_3] - u\} \cup \{v\}$ be arbitrary (v is given in (6)). By (5) and (6), $V(F_e) \cap V_3 = \emptyset = V(F_f) \cap V_3$, and $N_T(F_e) = V(e)$ and $N_T(F_f) = V(f)$. So $F_e \cap F_f = \emptyset$ when $e \ne f$. Hence, since there are $m - \delta$ edges in $G[V_3] - u$ and $V(F_e) \cap B \ne \emptyset \ne V(F_f) \cap B$ (by (5) and (6)), it follows from (4) that

$$\delta \ge |B| \ge |\mathcal{S}| = m - \delta + 1.$$

So $\delta \ge 1$ and $m \le 2\delta - 1$. Since $m \ge t\delta/2$, $t = 2$ or $t = 3$.

If $t = 2$ then $\delta = |B| = 1$, and $G[V_3] \cong K_2$; so by Lemma 4.1(5) and by (4), $k \ge |V_2| = |B| + |A| \ge 1 + (k-1) = k$, and (7) holds.

Thus we may assume $t = 3$. Then $\delta \geq m - \delta + 1 \geq 3\delta/2 - \delta + 1$. Therefore, $\delta = 2$, $G[V_3] \cong K_3$, and $|B| = 2$. Assume $V_3 = \{u, v, w\}$ and $B = \{b_1, b_2\}$. By (5) and (6), we may assume that $b_1 \in F_v$ and $b_2 \in F_{vw}$ and, by Lemma 4.1(2), that $b_1v, b_2v, b_2w \in E(T)$. Then $T - w$ is connected (as $N_T(w) \subseteq N_T(\{u, v\})$, contradicting (1) and completing the proof of (7).

We may assume

(8) $|V_3| = 1$.

For, suppose $|V_3| \geq 2$. Then by (7), $G[V_3] \cong K_2$. So let $V_3 = \{u, v\}$. If $V_2 \subseteq N(u)$ then $T - v$ is connected (as $N_T(v) \subseteq V_2 \cup \{u\}$ by Lemma 4.1(2)), contradicting (1). Thus $V_2 \not\subseteq N(u)$. Therefore, since $|N(u) - T| \leq 2$ (by (3)) and $|V_2| = k$ (by (7)), it follows from $|N(u)| \geq k + 2$ that $x \in N(u)$.

Similarly, we have $x \in N(v)$. Then $xuvx$ is a cycle in $G - X$ (as $X \subseteq V_1 \cup V_2$) by Lemma 4.1(3). Since $k \geq 3$, $G - \{u, v, x\}$ is $(k-1)$-connected and contains X; so the assertion of the lemma holds. Thus we may assume (8).

(9) $N_T(u) = V_2$, $|N(u) - V(T)| = 2$, $|V(D_r)| \geq 2$, and $N_T[u] = N(x) \cap V(T) = V_2 \cup V_3$.

Since $|N(u)| \geq k + 2$ and $N_T(u) \subseteq V_2$ (by Lemma 4.1(2)), it follows from (3) and (7) that $N_T(u) = V_2$ and $|N(u) - V(T)| = 2$. Since $N(x) \cap V(T) \subseteq V_2 \cup V_3$ (by definition of x-partition) and $|N(x) \cap V(D_r)| \leq 1$, we see that $V_2 \subseteq N(x)$, $|V(D_r)| \geq 2$, and $N_T[u] = N(x) \cap V(T) = V_2 \cup V_3$. This proves (9).

Let $V_2 = \{v_1, \ldots, v_k\}$. By (7) and Lemma 4.1(5), let P_i, for each $1 \leq i \leq k$, be the path which is the component of $G[V_1 \cup V_2] - E(G[V_2])$ containing v_i, and let $X = \{x_1, \ldots, x_k\}$ such that $x_i \in V(P_i)$. Since $|V(D_r)| \geq 2$, there exists $y \in V(D_r - x)$ with degree 1 in D_r. Then

(10) $u \in N(y)$ and $|N(y) \cap V_2| \geq k - 1$.

Consider the y-partition V_1', V_2', V_3' of $V(T)$. Then (3)-(9) holds for V_1', V_2', V_3'. In particular, $|V_2'| = k$ and $|V_3'| = 1$. Let $V_3' = \{u'\}$. Then by (9), $N_T(u') = V_2'$, $|N(u') - V(T)| = 2$, $N_T[u'] = N(y) \cap V(T) = V_2' \cup V_3'$.

If $u' = u$, then $V_2' = N_T(u') = N_T(u) = V_2$, $u \in N(y)$ and $|N(y) \cap V_2| = |N(y) \cap V_2'| = |V_2'| = k > k - 1$. So assume $u' \neq u$.

Since $G[V_1 \cup V_2] - E(G[V_2])$ is the disjoint union of $P_1, \ldots, P_k$, every vertex in $V_1 - V_2$ has degree at most 2 in T. Since u' has degree at least $k \geq 3$ in T, $u' \in V_2$. Without loss of generality, let $u' = v_1$. Then $v_1 \notin X$; so let v_1' be the neighbor of $u' = v_1$ on P_1. Then $N_T[u'] \subseteq V_2 \cup \{v_1', u\}$. Thus

$$|N(y) \cap V_2| \geq |N(y) \cap N_T[u']| - 2 = |N(y) \cap V(T)| - 2 = |V_2' \cup V_3'| - 2 = (k+1) - 2 = k - 1,$$

and $u \in N(y)$ (since $u \in N_T(u') \subseteq N(y)$). So we may assume (10).

From (10) and without loss of generality, we may assume that $v_1, \ldots, v_{k-1} \in N(y)$. Let $F = V(D_r) - \{x, y\}$. We may assume

(11) $N(F) \cap V_1 = \emptyset$.

For, suppose that there exists $v_i' \in V(P_i) \cap V_1$ such that $v_i' \in N(F)$ for some $1 \le i \le k$. If $i = k$ then $v_k \notin X$, xuv_kx is a cycle and, since $v_1, \ldots, v_{k-1} \in N(y)$, $G[V(T) \cup V(D_r)] - \{u, v_k, x\}$ is a connected subgraph of $G - \{u, v_k, x\}$ containing X; so the assertion of the lemma holds. If $i < k$ then $v_i \notin X$, yuv_iy is a cycle and, since $N(x) \cap V(T) = V_2 \cup V_3$ (by (9)), $G[V(T) \cup V(D_r)] - \{u, v_i, y\}$ is a connected subgraph of $G - \{u, v_i, y\}$ containing X; again the assertion of the lemma holds. So we may assume (11).

By (11), $N(F \cup \{x, u\}) = V_2 \cup \{y\}$ which has $k+1$ vertices. Since G is $(k+2)$-connected, $V_2 \cup \{y\}$ cannot be a cut in G; so $V(G) = V(D_r) \cup V_2 \cup \{u\}$, and $|V(T)| = |V_2 \cup V_3| = k+1$. Hence $X = V_2$. Moreover, $V(T) \subseteq N(y)$ (as $|N(y)| \ge k+2$ and $|N(y) - V(T)| = 1$).

Since $|N(u) - V(T)| \le 2$, we see that D_r has exactly two vertices of degree 1, and hence D_r is a path between x and y. Furthermore, $u \notin N(F)$ (by (3)), and therefore each vertex in F is adjacent to all of V_2. Finally, note that $G[X] = G[V_2]$ is not connected; as otherwise, $T - u = G[X]$ contradicts the choice of T in (1). So G is a k-fold wheel with center $G[X]$. ∎

Theorem 1.4 follows from the following result.

Theorem 4.3. *Let G be a $(k+2)$-connected graph, where $k \ge 2$, and let $X \subseteq V(G)$ with $|X| = k$. Then either G is a k-fold wheel with center $G[X]$, or there exists an induced cycle C in $G - X$ such that $G - V(C)$ is connected.*

Proof. Suppose G is not a k-fold wheel with center $G[X]$. By Lemma 4.2, there is a connected subgraph T of G containing X and there is an induced cycle C in $G - V(T)$. So T is contained in some component of $G - V(C)$, say U_0. Let $U_1, \ldots, U_r$ be the components of $G - V(C) - V(U_0)$. We may select C and T so that $S(C) := (|U_0|, |U_1|, \ldots, |U_r|)$ is maximal with respect to the lexicographic ordering.

If $r = 0$ then C is the desired cycle. So we may assume $r \ge 1$. Since G is $(k+2)$-connected, U_r has at least $k+2$ neighbors on C. Choose $u, v \in N(U_r) \cap V(C)$ such that there is an u-v path P in C whose internal vertices have no neighbor in U_r. Let Q be an induced path in $G[V(U_r) \cup \{u, v\}]$ between u and v. Then $C' = P \cup Q$ is an induced cycle in G. Since G is $(k+2)$-connected and C is induced, $V(C) - V(P)$ has at least one neighbor in $V(G) - V(C \cup U_r)$. Thus $S(C')$ is larger than $S(C)$, a contradiction. ∎

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