Subdivisions of K_5 in graphs containing $K_{2,3}$

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Abstract

Seymour and, independently, Kelmans conjectured that every 5-connected nonplanar graph contains a subdivision of K_5. We prove this conjecture for graphs containing $K_{2,3}$. As a consequence, the Kelmans-Seymour conjecture is true if the answer to the following question of Mader is affirmative: Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a subdivision of K_5?

1 Introduction

We follow the notation and terminology used in [10, 11]. In particular, for a given graph K we use TK to denote a subdivision of K. The vertices of a TK corresponding to the vertices of K are called the *branch* vertices of this TK. Hence the degree 4 vertices in a TK_5 are its branch vertices. A *separation* in a graph G is a pair (G_1, G_2) of subgraphs of G such that $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, and $E(G_i) \cup (V(G_i) - V(G_{3-i})) \neq \emptyset$ for $i = 1, 2$. If, in addition, $|V(G_1 \cap G_2)| = k$ then (G_1, G_2) is said to be a *k-separation*. A collection of paths is said to be *independent* if no vertex of any path is internal to any other path in the collection.

Mader [12] proved that every simple graph on $n \geq 5$ vertices and with at least $3n - 5$ edges contains TK_5, establishing a conjecture of Dirac [4]. In [7, 8], Dirac's conjecture is reduced to the following conjecture of Seymour [15]: Every 5-connected nonplanar graph contains TK_5.

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Kelmans [7] independently made the same conjecture two years later. In [10,11], the Kelmans-Seymour conjecture is established for graphs containing K_4^- (the graph obtained from K_4 by removing an edge).

Theorem 1.1 (*Ma and Yu*). *Every 5-connected nonplanar graph containing K_4^- contains TK_5.*

One important step in [10] is to deal with the case when a 5-connected nonplanar graph G admits a 5-separation (G_1, G_2) such that $|V(G_2)| \geq 7$ and G_2 has a plane representation in which all vertices in $V(G_1 \cap G_2)$ are incident with a common face. It is shown in [10] that in G_2 one can find a special collection of independent paths (used to construct a TK_5 in G). This result is also used in [5] by Krakovski, Stephens and Zha to show that the Kelmans-Seymour conjecture holds for graphs embedded in any surface (other than the sphere) with representativity at least 5.

It turns out to be very useful to exclude K_4^-. For example, by working with K_4^--free graphs, Kawarabayashi [6], Aigner-Horev [1], and Ma, Thomas and Yu [9] independently proved the Kelmans-Seymour conjecture for apex graphs. (A graph is said to be *apex* if it has an *apex vertex*, i.e., a vertex whose deletion results in a planar graph.) In this paper we prove the Kelmans-Seymour conjecture for graphs containing $K_{2,3}$, and our proof makes heavy use of the fact that we can assume the graphs to be K_4^--free.

Theorem 1.2 *Every 5-connected nonplanar graph containing $K_{2,3}$ contains TK_5.*

Theorems 1.1 and 1.2 imply that the Kelmans-Seymour conjecture holds if the answer to the following question of Mader [12] is affirmative: Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a TK_5?

In order to give a high level description of our proof of Theorem 1.2, we need additional notation and terminology. Let H be a graph and $A \subseteq V(H)$. We use $H[A]$ to denote the subgraph of H induced by A, and use $N_H(A)$ to denote the neighborhood of A. For any subgraph K of H, we let $H[K] := H[V(K)]$ and $N_H(K) := N_H(V(K))$. When understood, the subscript H may be omitted. For any positive integer k, we say that H is (k, A)-*connected* if, for any cut set T of H with $|T| \leq k-1$, each component of $H - T$ contains a vertex in A.

We now introduce a concept that is closely related to existence of disjoint paths. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- (a) for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,
- (b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and
- (c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane with no edge crossings.

If, in addition, $b_0, b_1, \ldots, b_n$ are vertices in G such that $b_i \notin A$ for $0 \leq i \leq n$ and $A \in \mathcal{A}$, $p(G, A)$ can be drawn in a closed disc with no edge crossings, and $b_0, b_1, \ldots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_0, b_1, \ldots, b_n)$ is 3-planar.

We make a simple, but useful, observation. Suppose for any $A \in \mathcal{A}$ and for any $u, v \in N(A)$, $G[A \cup \{u, v\}]$ has a path from u to v. If P is a path in $p(G, \mathcal{A})$ then we may produce a path P^* in G with the same ends of P as follows: For each edge uv of P with $\{u, v\} \subseteq N(A_i)$ for some i, replace uv with a path in $G[A_i \cup \{u, v\}]$ between u and v. As a consequence, any set of independent paths in $p(G, \mathcal{A})$ gives a set of independent paths in G with the same ends.

Given a graph G and $S \subseteq V(G)$, we say that (G, S) is *planar* if G has a drawing in a closed disc in the plane without edge crossings such that the vertices in S all appear on the boundary of the disc; in which case, the boundary of the unbounded face of G is called the *outer walk* of G (and *outer cycle* if it is a cycle). We say that (G, S) is *3-planar* if the vertices in S can be ordered as $b_0, \ldots, b_n$ such that $(G, b_0, \ldots, b_n)$ is 3-planar.

Another concept we need is from [3]. A *block* of a graph G is either a maximal 2-connected subgraph of G or a subgraph of G induced by a cut edge. A block is *nontrivial* if it is 2-connected, and it is *trivial* otherwise. A connected graph C is a *chain* if its blocks can be labeled as $B_1, \ldots, B_k$, where $k \geq 1$ is an integer, and its cut vertices can be labeled as $v_1, \ldots, v_{k-1}$ such that

(i) $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for $1 \leq i \leq k-1$ and

(ii) $V(B_i) \cap V(B_j) = \emptyset$ for $1 \leq i, j \leq k$ with $|i - j| \geq 2$.

We write $C := B_1 v_1 B_2 v_2 \ldots v_{k-1} B_k$ for this situation, and also view C as $\bigcup_{i=1}^{k} B_i$. If $k \geq 2$, $v_0 \in V(B_1) - \{v_1\}$ and $v_k \in V(B_k) - \{v_{k-1}\}$, or, if $k = 1$, $v_0, v_k \in V(B_1)$ and $v_0 \neq v_k$, then we say that C is a v_0-v_k *chain* or a chain *from* v_0 *to* v_k, and we denote this by $C := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$.

Let G be a graph and let $C := v_0 B_1 v_1 \ldots v_{k-1} B_k v_k$ be a chain from v_0 to v_k. If C is an induced subgraph of G, then we say that C is a *chain in* G. We say that C is a *planar chain in* G if, for each $1 \leq i \leq k$ with $|V(B_i)| \geq 3$ (or equivalently, B_i is 2-connected), there exist distinct vertices $x_i, y_i \in V(G) - V(C)$ such that

- $N(B_i - \{v_{i-1}, v_i\}) = \{v_{i-1}, v_i, x_i, y_i\}$,
- $(G[V(B_i) \cup \{x_i, y_i\}] - x_i y_i, x_i, v_{i-1}, y_i, v_i)$ is planar, and
- $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{x_i, y_i, v_{i-1}, v_i\}$.

Note that $\{x_i, y_i\} \cap \{x_j, y_i\}$ may be nonempty. We say that C is a *3-planar* chain if in the above definition of a planar chain we allow $x_i = y_i$ and replace the second condition by "if $x_i \neq y_i$ then $(G[V(B_i) \cup \{x_i, y_i\}] - x_i y_i, x_i, v_{i-1}, y_i, v_i)$ is 3-planar".

We are now ready to give a high level description of our proof of Theorem 1.2. Let G be a 5-connected graph and $\{x_1, x_2, y_1, y_2, y_3\} \subseteq V(G)$ such that $G[x_1, x_2, y_1, y_2, y_3] \cong K_{2,3}$ in which x_1, x_2 have degree 3. We will force a K_4^- in G and invoke Theorem 1.1, or force a 5-separation (G_1, G_2) such that G_2 is apex with apex vertex a and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar, and then invoke Corollary 2.9 proved in Section 2. For subgraphs H, K of G, we use $H - K$ to denote $H - V(H \cap K)$.

Step 1. We show that either G contains TK_5 or $H := G - \{y_1, y_2, y_3\}$ contains a 3-planar chain from x_1 to x_2, say C, such that $H - C$ is 2-connected. This is done by first producing an induced path X in H between x_1 and x_2 such that $H - X$ is connected, then augment a given 2-connected block in $H - X$. In the case the given block cannot be augmented we find a TK_5 or are left with the desired 3-planar chain. This is dealt with in Section 3.

STEP 2. There are two types of blocks in a 3-planar chain. In Section 4, we show that if there is a block, say D, with two neighbors in $H - C$, say b_D, c_D, then G contains TK_5. This is done roughly as follows. Let D^* be obtained from $G[D + \{b_D, c_D, y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 to a single vertex y, and let u_D, v_D be the ends of D. Then D^* is an apex graph with apex vertex y, and $(D^* - y, b_D, u_D, c_D, v_D)$ is 3-planar. We first show that G contains TK_5 or D^* is $(5, \{b_D, c_D, u_D, v_D, y\})$-connected. We then prove two results in Section 2 which in turn allow us to find a special collection of independent paths in D^*. Finally, we use these paths to force a 5-separation (G_1, G_2) in G such that G_2 is apex with apex vertex a and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar, and invoke Corollary 2.9.

STEP 3. We may thus assume that each 2-connected block of C has only one neighbor in $H - C$. We show that at least two of $\{y_1, y_2, y_3\}$ have neighbors in $H - C$. This makes it easier to find a TK_5. Again, whenever we are stuck we are rescued by a K_4^- or a 5-separation (G_1, G_2) in which G_2 is apex with apex vertex a and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. This is done in Section 5.

STEP 4. Finally, we arrive at the case when C is simply an induced path X. It is then easy to show that G contains TK_5 or none of $\{y_1, y_2, y_3\}$ has a neighbor in $X - \{x_1, x_2\}$. So $G - X$ is 2-connected. If in $G - X$ there is a cycle containing $\{y_1, y_2, y_3\}$ then such a cycle, together with $G[\{x_1, x_2, y_1, y_2, y_3\}] \cup X$, gives a TK_5 in G. So we may assume that such a cycle does not exist in $G - X$. Then we know the structure of $G - X$, given by a result of Watkins and Mesner in [21]. A case analysis similar to that in [10] finds a TK_5 in G.

2 Previous results and lemmas

In this section we list some known results and prove a few lemmas that are needed in our proof of Theorem 1.2. We begin with a result of Tutte [20].

Lemma 2.1 (*Tutte*). *Let G be a 3-connected graph, $e \in E(G)$ and $v \in V(G)$ such that v is not incident with e. Then $G - v$ contains an induced cycle C such that $e \in E(C)$ and $G - C$ is connected.*

We will need the following result of Seymour [16] about the existence of disjoint paths; equivalent versions can be found in [14, 17, 19].

Lemma 2.2 (*Seymour*). *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is 3-planar.*

We state a simpler version for graphs with higher connectivity.

Corollary 2.3 *Let G be a connected graph and s_1, s_2, t_1, t_2 be distinct vertices of G such that G is $(4, \{s_1, s_2, t_1, t_2\})$-connected. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is planar.*

We will heavily use the $k = 3$ case of the following result of Perfect [13].

Lemma 2.4 (*Perfect*) *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G-u)$. Suppose there exist k independent paths from u to distinct $a_1, \ldots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \ldots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \ldots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i = 1, \ldots, k$.*

We also need a result of Watkins and Mesner [21] on cycles through three vertices.

Lemma 2.5 (*Watkins and Mesner*). *Let R be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of R. Then there is no cycle through y_1, y_2 and y_3 in R if, and only if, one of the following statements holds.*

(*i*) *There exists a 2-cut S in R and, for $u \in \{y_1, y_2, y_3\}$, there exist pairwise disjoint subgraphs D_u of $R - S$ such that $u \in V(D_u)$ and each D_u is a union of components of $R - S$.*

(*ii*) *For $u \in \{y_1, y_2, y_3\}$, there exist 2-cuts S_u in R and pairwise disjoint subgraphs D_u of R, such that $u \in V(D_u)$, each D_u is a union of components of $R - S_u$, $S_{y_1} \cap S_{y_2} \cap S_{y_3} = \{z\}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*

(*iii*) *For $u \in \{y_1, y_2, y_3\}$, there exist pairwise disjoint 2-cuts S_u in R and pairwise disjoint subgraphs D_u of $R - S_u$ such that $u \in V(D_u)$, D_u is a union of components of $R - S_u$, and $R - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_u.*

The lemmas above are used in [10,11] to prove Theorem 1.1, which turns out to be useful here as well. The following lemma is proved in [10] and also needed here.

Lemma 2.6 (*Ma and Yu*) *Let G be a 5-connected nonplanar graph, and let (G_1, G_2) be a 5-separation of G such that $|V(G_2)| \geq 7$ and $(G_2, V(G_1) \cap V(G_2))$ is planar. Then G contains TK_5.*

In order to prove Theorem 1.2, we need to generalize Lemma 2.6 by allowing G_2 to be apex. Our original work on this generalization is quite complex, which is simplified by the following lemma (and its proof) due to Thomas [18]. For a vertex x in a graph H, we use $d_H(x)$ to denote the degree of x in H.

Lemma 2.7 (*Thomas*) *Let G be a connected graph with $|V(G)| \geq 7$, let $A \subseteq V(G)$ with $|A| = 5$, and let $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is planar, and either (1) $A - \{a\}$ is independent and $d_{G-a}(v) \geq 2$ for all $v \in A - \{a\}$ or (2) $A - \{a\}$ is not independent and $d_{G-a}(v) \geq 4$ for all $v \in A - \{a\}$. Then G contains K_4^-, or G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$.*

Proof. Let $A = \{a, a_1, a_2, a_3, a_4\}$, and assume that $G - a$ is drawn in a closed disc in the plane without edge crossings such that a_1, a_2, a_3, a_4 occur on the boundary of the disc in clockwise order. Since $|V(G)| \geq 7$ and G is $(5, A)$-connected, $a_1a_3, a_2a_4 \notin E(G)$.

Let $H = (G - a) + \{a_1a_2, a_2a_3, a_3a_4, a_4a_1\}$ if (1) holds, and let $H = G - a$ if (2) holds; so that when (1) occurs H is a plane graph with outer cycle $a_1a_2a_3a_4a_1$. Note that the minimum degree of H satisfies $\delta(H) \geq 4$. Since G is $(5, A)$-connected, for $v \in V(H) - \{a_1, a_2, a_3, a_4\}$, if $d_H(v) = 4$ then $va \in E(G)$. If G contains K_4^- we are done. So we may assume that $K_4^- \not\subseteq G$.

Let $uvwu$ be a facial triangle in H. We say that $uvwu$ (and the face it bounds) is *bad* if $|\{u, v, w\} \cap A| = 2$, or $\{u, v, w\} \cap A = \{a_i\}$ for some $1 \leq i \leq 4$ with $d_H(a_i) = 4$. Since $K_4^- \not\subseteq G$, there are at most 8 bad facial triangles in H. In fact, it is easy to show that if there are 8 bad facial triangles in H then $a_1a_2a_3a_4a_1$ is the outer cycle of H, the outer cycle of $H - \{a_1, a_2, a_3, a_4\}$ is a 4-cycle $b_1b_2b_3b_4$, and we may choose the notation so that $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ is a cycle in H. If $|V(G)| = 9$ then, since G is $(5, A)$-connected, $\{b_1, b_2, b_3, b_4\} \subseteq N_G(a)$, or $b_1b_3 \in E(G)$, or $b_2b_4 \in E(G)$; so G contains K_4^-, a contradiction. If $|V(G)| = 10$ then, since G is $(5, A)$-connected, the vertex in $V(G) - \{a, a_i, b_i : i = 1, 2, 3, 4\}$ is adjacent to all of $\{b_1, b_2, b_3, b_4\}$, forcing a K_4^- in G, a contradiction. So $|V(G)| \geq 11$, then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, b_1, b_2, b_3, b_4\}$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$. Thus, we may assume that H has at most 7 bad facial triangles.

We may assume that if $uvwu$ is a facial triangle and is not bad, then two of $\{u, v, w\}$ must have degree at least 5 in H. Clearly $\{u, v, w\} \not\subseteq A$ because $a_1a_3, a_2a_4 \notin E(G)$. Now let $v, w \notin A$. If $d_H(v) \geq 5$ and $d_H(w) \geq 5$ then we are done. So we may assume that $d_H(v) = 4$; hence $va \in E(G)$. If $d_H(w) = 4$ then $wa \in E(G)$ and $G[\{a, u, v, w\}]$ contains K_4^-, a contradiction. So $d_H(w) \geq 5$. Similar argument shows that if $u \notin A$ then $d_H(u) \geq 5$. So assume $u \in A$. Then $d_H(u) \geq 5$ as $uvwu$ is not bad.

We now derive a contradiction by applying a simple discharging to H. Let $F(H)$ denote the set of faces of H, and for any $f \in F(H)$ let $d_H(f)$ denote the number of vertices incident with f. Let $\sigma : V(H) \cup F(H) \to \mathbb{Z}$ (the set of integers) such that $\sigma(x) = 4 - d_H(x)$ for all $x \in V(H) \cup F(H)$. Then by Euler's formula, the total charge is

$$\sigma(H) = \sum_{v \in V(H)} \sigma(v) + \sum_{f \in F(H)} \sigma(f) = 8.$$

Note that for any $x \in V(H) \cup F(H)$, if $\sigma(x) > 0$ then $x \in F(H)$, $d_H(x) = 3$, and $\sigma(x) = 1$. We now redistribute charges as follows, such that the total charge remains unchanged. For each $f \in F(H)$ with $d_H(f) = 3$ and f not bad, pick two of its incident vertices with degree at least 5 in H, and send a charge 1/2 from f to each of these two vertices. Let τ denote the resulting charge function. Then $\tau(f) \leq 0$ for all $f \in F(H)$ that is not bounded by a triangle or is not bad, and $\tau(x) = 0$ if $x \in V(H)$ and $d_H(x) = 4$. Now suppose $x \in V(H)$ and $d_H(x) \geq 5$. Since $K_4^- \not\subseteq G$, x is contained in at most $\lfloor d_H(x)/2 \rfloor$ facial triangles. Hence $\tau(x) \leq \sigma(x) + \lfloor d_H(x)/2 \rfloor/2 = 4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2$. Note that

$$4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2 = \begin{cases} 4 - 3k, & \text{if } d_H(x) = 4k; \\ 3 - 3k, & \text{if } d_H(x) = 4k + 1; \\ 5/2 - 3k, & \text{if } d_H(x) = 4k + 2; \\ 3/2 - 3k, & \text{if } d_H(x) = 4k + 3. \end{cases}$$

Since $d_H(x) \geq 5$, we have $k \geq 1$ and $k \geq 2$ if $d_H(x) = 4k$. Hence, $\tau(x) \leq 4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2 \leq 0$. Thus the total new charge is $\tau(H) \leq 7$ because there are at most 7 bad facial triangles. This is a contradiction. ∎

The following is an easy consequence of Lemma 2.7. It was proved independently by Kawarabayashi [6], by Aigner-Horev [1], and by Ma, Thomas and Yu [9].

Corollary 2.8 *Every 5-connected nonplanar apex graph contains TK_5.*

Proof. Let G be a 5-connected nonplanar apex graph and a be its apex vertex. By Theorem 1.1, we may assume that $K_4^- \not\subseteq G$. So $G-a$ has a plane representation in which the outer cycle is not a triangle. Let a_1, a_2, a_3, a_4 be four arbitrary vertices in the outer cycle of $G-a$, and let $A = \{a, a_1, a_2, a_3, a_4\}$. Then G, A, a satisfy the conditions of Lemma 2.7 (in particular, (1) or (2) depending on whether or not A is independent).

Hence, by Lemma 2.7 and the assumption that $K_4^- \not\subseteq G$, G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$. We choose such (G_1, G_2) so that G_2 is minimal, and let $A' = V(G_1 \cap G_2)$.

If $|V(G_2)| = 7$ then, since G_2 is $(5, A')$-connected and $(G_2 - a, A' - \{a\})$ is planar, $K_4^- \subseteq G_2$, a contradiction. So $|V(G_2)| \geq 8$. Hence, by the minimality of G_2, A' is independent in G_2 and $d_{G_2-a}(v) \geq 2$ for all $v \in A' - \{a\}$. So G_2, A', a satisfies the conditions of Lemma 2.7 (in particular, (1)). As a consequence of Lemma 2.7, $K_4^- \subseteq G_2$, a contradiction. ∎

As mentioned before, we need an apex version of Lemma 2.6, which is also an easy consequence of Lemma 2.7.

Corollary 2.9 *Let G be a 5-connected nonplanar graph, (G_1, G_2) a 5-separation of G, and $a \in A := V(G_1) \cap V(G_2)$ such that $|V(G_2)| \geq 7$ and $(G_2 - a, A - \{a\})$ is planar. Then G contains TK_5.*

Proof. We choose such separation (G_1, G_2) so that G_2 is minimal. Then $A - \{a\}$ is independent in G_2. If $|V(G_2)| = 7$ then, since G_2 is $(5, A)$-connected and $(G_2 - a, A - \{a\})$ is planar, $K_4^- \subseteq G_2$. If $|V(G_2)| \geq 8$ then by the minimality of G_2, A is independent in G and $d_{G_2-a}(v) \geq 2$ for all $v \in A - \{a\}$; so $K_4^- \subseteq G_2$ by Lemma 2.7. Therefore, the assertion of this corollary follows from Theorem 1.1. ∎

In the proof of Lemma 2.6 in [10], an important step is to find a collection of independent paths in G_2, the planar side. For the purpose of this paper, we need to extend this to the apex side of a 5-separation. The following result is due to Thomas [18] which significantly simplifies our proofs of such results (see Corollaries 2.11 and 2.12).

Lemma 2.10 (*Thomas*). *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is planar, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Let $w \in V(G) - A$ and assume that the vertices in $G-a$ cofacial with w induce a cycle C in $G-a$. Then there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C)| \leq 1$ and $|V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$.*

Proof. Since G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$, A must be independent in G. Let $H := G - (C - N(w))$.

Suppose H has four paths P_1, P_2, P_3, P_4 from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$ and $|V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$. We may assume that these paths are induced paths. Hence $|V(P_i \cap C)| \leq 1$ for $1 \leq i \leq 4$. (Note that $|V(P_i) \cap C| = 0$ when $P_i = wa$.) So P_i, $1 \leq i \leq 4$, are the desired paths.

Thus we may assume that such paths in H do not exist. By Menger's theorem, there is a cut T, $|T| \leq 3$, in H separating w from A. For convenience, assume that $G - a$ is drawn in a closed disc in the plane with no edge crossings such that $A - \{a\}$ is contained in the boundary of the disc. Thus there is a simple closed curve γ in the plane intersecting $G - a$ only in $(T - \{a\}) \cup (V(C) - N(w))$ such that w is inside γ and $A - \{a\}$ is outside γ or on γ. When $|T - \{a\}| \geq 2$, the elements of $T - \{a\}$ divide γ into $|T - \{a\}|$ simple curves (including the points in $T - \{a\}$), called the *segments* of γ. When $|T - \{a\}| \leq 1$, we define γ to be its only segment. For two distinct points u, v on γ we use $u\gamma v$ to denote the simple curve in γ from u to v in clockwise order; and if $u = v$ then $u\gamma v$ consists of the single point $u = v$. We claim that

(1) if $u, v \in V(C) - N(w)$ and $u\gamma v$ is contained in a segment of γ, then $uCv - \{u, v\}$ contains no neighbor of w.

For, otherwise, we may choose such u, v that u and v are consecutive on γ. Then $\{a, u, v, w\}$ is a 4-cut in G separating $uCv - \{u, v\}$ from A, contradicting the $(5, A)$-connectedness of G.

Note that $\gamma \cap V(C) \cap N(w) \subseteq T - \{a\}$ and $T \cap (V(C) - N(w)) = \emptyset$. Also note that since G is $(5, A)$-connected,

(2) $|T| + |\gamma \cap (V(C) - N(w))| \geq 5$.

We consider cases based on $|T - \{a\}|$.

Case 1. $|T - \{a\}| \leq 1$.

In this case γ has only one segment; so any two vertices in $\gamma \cap V(C)$ are contained in a segment. First, suppose $T - \{a\} = \emptyset$. Then $|\gamma \cap (V(C) - N(w))| \geq 4$ by (2). Let $u, v \in \gamma \cap (V(C) - N(w))$. By (1), neither $uCv - \{u, v\}$ nor $vCu - \{u, v\}$ contains a neighbor of w. Hence, $\{a, u, v\}$ is a 3-cut in G separating w from A, a contradiction.

Now, suppose $|T - \{a\}| = 1$. Then $|\gamma \cap (V(C) - N(w))| \geq 3$ by (2). Let $u, v \in \gamma \cap (V(C) - N(w))$ such that $T - \{a\} \subseteq v\gamma u$ and, subject to this, $v\gamma u$ is minimal. Then by (1), $uCv - \{u, v\}$ contains no neighbor of w. So $\{a, u, v\} \cup (T - \{a\})$ is a cut in G separating w from A, a contradiction.

Case 2. $|T - \{a\}| = 2$.

Let $T - \{a\} = \{t_1, t_2\}$. Then $|\gamma \cap (V(C) - N(w))| \geq 2$ by (2).

First, assume $(t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C) = \emptyset$. Then for $i = 1, 2$, let $u_i \in (t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i. By (1), $N(w) \subseteq V(u_1Cu_2) \cup \{a\}$. Hence $\{a, t_1, t_2, u_1, u_2\}$ is a cut in G separating w and $N(w)$ from A, a contradiction (to the nonexistence of such a separation).

Thus $(t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C) \neq \emptyset$. Similarly, $(t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C) \neq \emptyset$.

For $i = 1, 2$, let $u_i \in (t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i, and $v_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with v_i closest to t_i. Then by (1), $N(w) \subseteq V(u_1Cv_1 - \{u_1, v_1\}) \cup V(v_2Cu_2 - \{u_2, v_2\}) \cup \{a\}$. As $|N(w) \cap V(C)| \geq 4$, we may assume by symmetry that $|N(w) \cap V(u_1Cv_1 - \{u_1, v_1\})| \geq 2$. Hence $\{a, t_1, u_1, v_1, w\}$ is a cut in G separating A from at least two vertices, a contradiction.

Case 3. $|T - \{a\}| = 3$.

Let $T - \{a\} = \{t_1, t_2, t_3\}$. In this case, $a \notin T$ and a has no neighbor strictly inside γ. By (2), $|\gamma \cap (V(C) - N(w))| \geq 2$.

First, assume $\gamma \cap (V(C) - N(w))$ is contained in some segment of γ, say $t_1\gamma t_2$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i. By (1), $N(w) \subseteq V(u_2Cu_1)$. Hence $\{t_1, t_2, t_3, u_1, u_2\}$ is a cut in G separating w and $N(w)$ from A, a contradiction.

Therefore, $\gamma \cap (V(C) - N(w))$ is not contained in any segment of γ.

Next, assume that the interior of some segment of γ, say $t_3\gamma t_1 - \{t_1, t_3\}$, is disjoint from $V(C)$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i; and for $i = 2, 3$, let $v_i \in (t_2\gamma t_3 - \{t_2, t_3\}) \cap V(C)$ with v_i closest to t_i. Then by (1), $N(w) \subseteq V(u_2Cv_2 - \{u_2, v_2\}) \cup V(v_3Cu_1 - \{u_1, v_3\})$. Since $|N(w) \cap V(C)| \geq 4$, $|N(w) \cap V(u_2Cv_2 - \{u_2, v_2\})| \geq 2$ or $|N(w) \cap V(v_3Cu_1 - \{u_1, v_3\})| \geq 2$. In the first case, $\{t_2, u_2, v_2, w\}$ is cut in G separating A from some neighbor of w, a contradiction. In the second case, $\{t_1, t_3, u_1, v_3, w\}$ is a cut in G separating A from at least two vertices, a contradiction.

Thus, $(t_i\gamma t_{i+1} - \{t_i, t_{i+1}\}) \cap (V(C) - N(w)) \neq \emptyset$ for $i = 1, 2, 3$, where $t_4 = t_1$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i; for $i = 2, 3$, let $v_i \in (t_2\gamma t_3 - \{t_2, t_3\}) \cap V(C)$ with v_i closest to t_i; and for $i = 1, 3$, let $w_i \in (t_3\gamma t_1 - \{t_1, t_3\}) \cap V(C)$ with w_i closest to t_i. Then by (1), $N(w) \subseteq V(u_2Cv_2 - \{u_2, v_2\}) \cup V(v_3Cw_3 - \{v_3, w_3\}) \cup V(w_1Cu_1 - \{u_1, w_1\})$. Since $|N(w) \cap V(C)| \geq 4$, $|N(w) \cap V(u_2Cv_2 - \{u_2, v_2\})| \geq 2$ or $|N(w) \cap V(v_3Cw_3 - \{v_3, w_3\})| \geq 2$ or $|N(w) \cap V(w_1Cu_1 - \{u_1, w_1\})| \geq 2$. In the first case, $\{t_2, u_2, v_2, w\}$ is a cut in G separating A from some neighbor of w, a contradiction. In the second case, $\{t_3, v_3, w_3, w\}$ is a cut in G separating A from some neighbor of w, a contradiction. In the third case, $\{t_1, u_1, w_1, w\}$ is a cut in G separating A from some neighbor of w, a contradiction. ∎

As consequences of Lemma 2.10, we derive the following two results about independent paths.

Corollary 2.11 *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that G is $(5, A)$-connected, $(G - a, A - a)$ is planar, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Let $w \in N(a)$ such that w does not belong to the outer walk of $G - a$. Then*

- *(i) the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$,*
- *(ii) $G - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 3$.*

Proof. Let D denote the outer walk of $G - a$. Then $A - \{a\} \subseteq V(D)$ and $w \notin V(D)$. Since G is $(5, A)$-connected and by planarity of $G - a$, the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$. Applying Lemma 2.10, we obtain four paths P_1, P_2, P_3, P_4 with one of them, say P_4, being wa. Now P_1, P_2, P_3 are the desired paths. ∎

The next consequence of Lemma 2.10 is more technical. We require that $G - a$ be K_4^--free instead of G. This is because in certain applications of this corollary, the vertex a is the result of identifying several vertices and therefore may be contained in some K_4^-.

Corollary 2.12 *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$, such that $K_4^- \not\subseteq G - a$, G is $(5, A)$-connected, $(G - a, (A - a) \cup N(a))$ is planar, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Then $G - a$ is 2-connected. Moreover, either G is the graph obtained from the edge-disjoint union of an 8-cycle $x_1x_2x_3x_4x_5x_6x_7x_8x_1$ and a 4-cycle $x_2x_4x_6x_8x_2$ by adding a and the edges ax_i, $i = 2, 4, 6, 8$, with $A = \{a, x_1, x_3, x_5, x_7\}$, or there exists $w \in V(G) - A$ such that*

- *(i) the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$ such that $C \cap D = \emptyset$, where D denotes the outer cycle of $G - a$,*

(ii) *there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$, and*

(iii) *either $a \notin \bigcup_{i=1}^4 V(P_i)$, or $a \in \bigcup_{i=1}^4 V(P_i)$ and we may write $A - \{a\} = \{a_1, a_2, a_3, a_4\}$ such that $a \in V(P_1)$, $a_i \in V(P_i)$ for $2 \leq i \leq 4$, and $a_1, a_2, a_3, V(P_1 \cap D), a_4$ occur on D in cyclic order.*

Proof. Since G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G_2)| \geq 7$,

(1) A is independent in G and every vertex in A has degree at least 2 in G.

We claim that

(2) $G - a$ is 2-connected.

Otherwise, we may write $G - a = H_1 \cup H_2$ such that $|V(H_1) \cap V(H_2)| \leq 1$ and $|V(H_i)| \geq 2$ for $i = 1, 2$. By (1), $V(H_i) - V(H_1 \cap H_2) \not\subseteq A$ for $i = 1, 2$. Note that $|V(H_i) \cap A| \leq 2$ for some i. Hence G has a separation (G_1, G_2) such that $G_2 - (V(G_1) \cap V(G_2)) = G[(H_i - H_{3-i}) \cup \{a\}]$ and $V(G_1 \cap G_2) = (V(H_i) \cap A) \cup V(H_1 \cap H_2) \cup \{a\}$ (which has size at most 4). Clearly, $A \subseteq V(G_1)$. Since $V(H_2) - V(H_1) \not\subseteq A$, $V(G_2) - V(G_1) \neq \emptyset$. This contradicts the assumption that G is $(5, A)$-connected.

By (2), let D denote the outer cycle of $G - a$; so $A - \{a\} \subseteq V(D)$. We claim that

(3) every edge in $(G - a) - E(D)$ with both ends on D must join two neighbors of a vertex in $A - \{a\}$.

Let $uv \in E(G - a) - E(D)$ with $u, v \in V(D)$. Then $G - a$ has a 2-separation (H_1, H_2) such that $V(H_1) \cap V(H_2) = \{u, v\}$ and $V(H_i) - V(H_{3-i}) \neq \emptyset$ for $i = 1, 2$. By symmetry, we may assume that $|V(H_1 - \{u, v\}) \cap A| \leq |V(H_2 - \{u, v\}) \cap A|$.

First, suppose $|V(H_1 - \{u, v\}) \cap A| = 2$. Then, by (1) and since G is $(5, A)$-connected, $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$ is a 5-cut in G separating A from just one vertex, say x, and x is adjacent to all of $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$. Then it is easy to see that $K_4^- \subseteq H_1 \subseteq G - a$, a contradiction.

Thus, $|V(H_1 - \{u, v\}) \cap A| \leq 1$. Since G is $(5, A)$-connected, $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$ cannot be a cut in G separating A from some vertex; so $|V(H_1)| = 3$ and the vertex in $V(H_1) - \{u, v\}$ must belong to A. So we have (3).

Suppose $V(G - a) = V(D)$. By (3) and because $(G - a, A - \{a\})$ is planar and G is $(5, A)$-connected, we see that G must be the graph obtained from the edge-disjoint union of an 8-cycle $x_1x_2x_3x_4x_5x_6x_7x_8x_1$ and a 4-cycle $x_2x_4x_6x_8x_2$ by adding a and the edges ax_i, $i = 2, 4, 6, 8$, with $A = \{a, x_1, x_3, x_5, x_7\}$.

So we may assume that $V(G - a) \neq V(D)$. We claim that

(4) there exists $w \in V(G - a) - V(D)$ such that w is not cofacial with any vertex of D.

For, suppose every vertex of $V(G - a) - V(D)$ is cofacial with some vertex of D. Then $G - a - V(D)$ is outerplanar. So there exists $w \in V(G - a) - V(D)$ such that w has degree at most 2 in $G - a - V(D)$.

Since G is $(5, A)$-connected and $N(a) \subseteq V(D)$, w has at least three neighbors in D. Let $w_1, \dots, w_k$ be the neighbors of w on D (so $k \geq 3$), and assume that they occur on D in this clockwise order. Moreover, by planarity, we may choose w so that there is no vertex inside the cycle ww_1Dw_kw. Since $K_4^- \not\subseteq G - a$, $|V(w_1Dw_k)| \geq 4$. So by (1), $V(w_1Dw_k - \{w_1, w_k\}) \not\subseteq A$.

Suppose for some $v \in V(w_1Dw_k - \{w_1, w_k\}) - A$, $v \notin N(w)$. By symmetry, assume $v \in V(w_1Dw_{k-1})$. Then since G is $(5, A)$-connected and by (3), there exist $vv_1, vv_2 \in E(G - a) - E(D)$ such that $\{v, v_i\} = N(a_i)$ for $a_i \in A$ $(i = 1, 2)$, and $N(v) = \{a, a_1, a_2, v_1, v_2\}$. Assume $v_1 \in V(w_1Dv_2)$. Now by (1), $\{a, v_1, v_2\} \cup (A \cap V(v_2Dv_1))$ is a 5-cut of G separating A from $\{w, w_k\}$, a contradiction.

So $V(w_1Dw_k - \{w_1, w_k\}) - A \subseteq N(w)$. Let $v \in V(w_1Dw_k - \{w_1, w_k\}) - A$. Since G is $(5, A)$-connected, there exists $vv' \in E(G - a) - E(D)$ with $v' \in V(D)$. By (3), $\{v, v'\} = N(a_i)$ for some $a_i \in A$. By (1), $v' \notin A$; so $v, v' \in N(w)$. Now $G[\{a_i, v, v', w\}] \cong K_4^-$, a contradiction. This completes the proof of (4).

Since G is $(5, A)$-connected and by planarity of $G - a$, we see that the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$. Then $C \cap D = \emptyset$ by (4).

By applying Lemma 2.10, there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$. If $a \notin \bigcup_{i=1}^4 V(P_i)$ then (iii) holds. So we may assume without loss of generality that $a \in V(P_1)$.

Let $A - \{a\} = \{a_1, a_2, a_3, a_4\}$ such that $a_i \in V(P_i)$ for $i = 2, 3, 4$, let w_i denote the neighbor of w in P_i for $i = 1, 2, 3, 4$, and let a' denote the neighbor of a in P_1. If there exists a permutation ijk of $\{2, 3, 4\}$ such that a_1, a_i, a_j, a', a_k occur D in cyclic order then (iii) holds. So we may assume, without loss of generality, that a_1, a', a_2, a_3, a_4 occur on D in clockwise order. Since $C \cap D = \emptyset$, $a_1Da' \cup a'P_1w$ contains a path P_1' from a_1 to w such that $V(P_1' \cap C) = \{w_1\}$. Now P_1', P_2, P_3, P_4 show that (iii) holds. ∎

3 3-Planar chains

Throughout the rest of this paper, let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2, y_3 \in V(G)$ be distinct such that $K := G[x_1, x_2, y_1, y_2, y_3] \cong K_{2,3}$ in which x_1, x_2 have degree 3. Let $H := G - \{y_1, y_2, y_3\}$.

In this section we will show that G contains TK_5 or H contains a 3-planar chain C from x_1 to x_2 such that $H - C$ is 2-connected. We need the concept of a bridge. Let J be a graph and $L \subseteq J$. An *L-bridge* of J is a subgraph of J induced by the edges of a component of $J - L$ and all edges from that component to L.

First, we prove a very useful lemma that G contains TK_5 or no vertex other than x_1 and x_2 may be adjacent to two of $\{y_1, y_2, y_3\}$.

Lemma 3.1 *Suppose $x_3 \in V(H) - \{x_1, x_2\}$ and $|N(x_3) \cap \{y_1, y_2, y_3\}| \geq 2$. Then G contains TK_5.*

Proof. Without loss of generality, we may assume that $x_3y_1, x_3y_2 \in E(G)$. Note the symmetry among x_1, x_2, y_1, y_2 and between x_3 and y_3.

If $G - \{x_3, y_3\}$ contains four independent paths from some $u \in V(G) - \{x_1, x_2, x_3, y_1, y_2, y_3\}$ to x_1, x_2, y_1, y_2, respectively, then these paths and $K \cup y_1x_3y_2$ form a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that such paths do not exist. Then

(1) G has a 5-separation (H_1, H_2) such that $\{x_3, y_3\} \subseteq V(H_1) \cap V(H_2)$, $u \in V(H_1) - V(H_2)$, and $\{x_1, x_2, y_1, y_2\} \subseteq V(H_2)$.

We choose (H_1, H_2) in (1) so that H_2 is minimal. Let $S := V(H_1 \cap H_2) - \{x_3, y_3\} = \{s_1, s_2, s_3\}$. We may assume that

(2) $S \not\subseteq \{x_1, x_2, y_1, y_2\}$.

For, suppose $S \subseteq \{x_1, x_2, y_1, y_2\}$. By symmetry we may assume that $x_1 \notin S$. By Menger's theorem, $H_2 - \{y_1, y_2, y_3\}$ contains two independent paths P_2, P_3 from x_1 to x_2, x_3, respectively. If $H_1 - y_3$ contains disjoint paths from x_2 to x_3 and from y_1 to y_2 then these paths and $(K - y_3) \cup y_1 x_3 y_2 \cup P_2 \cup P_3$ form a TK_5 in G with branch vertices x_1, x_2, x_3, y_1, y_2. So we may assume that such disjoint paths do not exist. Then by Corollary 2.3, $(H_1 - y_3, x_2, y_1, x_3, y_2)$ is planar. If $|V(H_1) - V(H_2)| \geq 2$ then, by Corollary 2.9, G contains TK_5. So we may assume that $|V(H_1) - V(H_2)| = 1$. Thus, since G is $(5, A)$-connected, the unique vertex in $V(H_1) - V(H_2)$ is adjacent to x_2, y_1, y_2; so G contains K_4^- and hence TK_5 by Theorem 1.1.

By (2) we may assume $s_1 \notin \{x_1, x_2, y_1, y_2\}$. We claim that

(3) $H_2 - \{x_3, y_3\}$ contains four paths S_i, $i = 0, 1, 2, 3$, from $\{x_1, x_2, y_1, y_2\}$ to s_i, respectively, where $s_0 = s_1$, such that $S_0 \cap S_1 = \{s_1\}$, and $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and $\{i, j\} \neq \{0, 1\}$.

Let H_2' be obtained from $H_2 - \{x_3, y_3\}$ by duplicating s_1, and use s_0 to denote the duplicate of s_1. (Hence, s_0 and s_1 have the same neighborhood in H_2'.) By the minimality of H_2 and by Menger's theorem, H_2' contains four disjoint paths S_i from $\{x_1, x_2, y_1, y_2\}$ to s_i, $i = 0, 1, 2, 3$, respectively. Note that S_1, S_2, S_3 are paths in $H_2 - \{x_3, y_3\}$. By identifying s_0 with s_1, we view S_0 as a path in $H_2 - \{x_3, y_3\}$ from s_1.

(4) We may assume that s_1 has a unique neighbor in $H_1 - \{s_2, s_3\}$, and denote it by u.

If $H_1 - \{x_3, y_3\}$ contains independent paths P_2, P_3 from s_1 to s_2, s_3, then $S_0 \cup S_1 \cup (P_2 \cup S_2) \cup (P_3 \cup S_3) \cup K \cup y_1 x_3 y_2$ is a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2. So we may assume that such paths do not exist. Then $H_1 - \{x_3, y_3\}$ has a cut vertex v separating s_1 from $\{s_2, s_3\}$. Since G is 5-connected, the v-bridge of $H_1 - \{x_3, y_3\}$ containing s_1 is induced by the edge $s_1 v$. Hence (4) holds.

(5) We may assume that there exist $b_0 \in S_0$ and $b_1 \in S_1$ such that in $H_2 - \{x_3, y_3\}$, $\{b_0, b_1, s_2, s_3\}$ separates s_1 from $\{x_1, x_2, y_1, y_2\}$.

To see this let H_2'' be obtained from $H_2 - \{x_3, y_3\}$ by duplicating s_1 twice and identifying s_2 and s_3 (also denote it by s_2), and let s_1', s_1'' denote the duplicates of s_1.

Suppose H_2'' contains four disjoint paths from $\{s_1, s_1', s_1'', s_2\}$ to $\{x_1, x_2, y_1, y_2\}$. Then $H_2 - \{x_3, y_3\}$ has four independent paths to $\{x_1, x_2, y_1, y_2\}$, three from s_1 and one from s_2 or s_3, say s_2. Thus, these four paths, $K \cup y_1 x_3 y_2$, and a path in $H_1 - \{x_3, y_3, s_3\}$ from s_1 to s_2 form a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2.

So we may assume that such four paths in H_2'' do not exist. Then H_2'' has a separation (R, R') such that $|V(R) \cap V(R')| \leq 3$, $\{s_1, s_1', s_1'', s_2\} \subseteq V(R)$, and $\{x_1, x_2, y_1, y_2\} \subseteq V(R')$.

Note that $V(R) \cap V(R') \neq \{s_1, s_1', s_1''\}$ by (3). Since s_1, s_1', s_1'' have the same neighborhood in H_2'', $s_1, s_1', s_1'' \notin V(R) \cap V(R')$. Also, $s_2 = s_3 \in V(R) \cap V(R')$; for otherwise, $V(R) \cap V(R')$ is a cut in $H - \{x_3, y_3\}$ separating H_1 from $\{x_1, x_2, y_1, y_2\}$, contradicting the minimality of H_2 (among those (H_1, H_2) satisfying (1)).

Thus, $(H_2 - \{x_3, y_3\}) - \{s_2, s_3\}$ has a cut $T := V(R \cap R') - \{s_2 = s_3\}$ separating s_1 from $\{x_1, x_2, y_1, y_2\}$, and $s_1 \notin T$ and $|T| \leq 2$. Since $s_1 \notin T$ and because of S_0 and S_1 in (3), $|T| = 2$; so letting $T = \{b_0, b_1\}$ with $b_0 \in S_0$ and $b_1 \in S_1$, we complete the proof of (5).

Let R^* denote the component of $(H_2 - \{x_2, x_3\}) - \{b_0, b_1, s_2, s_3\}$ containing s_1. Choose $\{b_0, b_1\}$ so that R^* is minimal.

(6) We may assume that $s_2, s_3 \notin N(R^*)$, and for any $w \in \{x_3, y_3\}$, $G[R^* + \{b_0, b_1, w\}]$ contains independent paths from s_1 to b_0, b_1, w, respectively.

First, assume that s_2 or s_3, say s_2, has a neighbor in R^*. Then by the minimality of R^*, $G[R^* + \{b_0, b_1, s_2\}]$ contains three independent paths from s_1 to b_0, b_1, s_2, respectively; and we may assume that $s_1 S_0 b_0$ and $s_1 S_1 b_1$ are two of them. Now these three paths, $S_0 \cup S_1 \cup S_2 \cup S_3 \cup K \cup y_1 x_3 y_2$, and a path in $H_1 - \{s_2, x_3, y_3\}$ from s_1 to s_3 form a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2.

So we may assume that R^* contains no neighbor of $\{s_2, s_3\}$. Since G is 5-connected and by (4), R^* has neighbors of both x_3 and y_3. If $R^* = \{s_1\}$ then $s_1 x_3, s_1 y_3 \in E(G)$; so (6) holds. Hence we may assume that $|V(R^*)| \geq 2$. By the minimality of R^*, we see that for any $w \in \{x_3, y_3\}$, $G[R^* + \{b_0, b_1, w\}]$ contains independent paths from s_1 to w, b_0, b_1, respectively. Again, we have (6).

Let $R_1 = G[R^* + \{b_0, b_1, u, x_3, y_3\}]$. Note that when $R^* \neq \{s_1\}$ we have symmetry between R_1 and H_1, since we will not need the minimality of H_2 in the remainder of the proof.

(7) We may assume that $|V(H_1)| \geq 7$.

For, suppose $|V(H_1)| = 6$. Then u (see (4)) is adjacent to all of $\{s_1, s_2, s_3, x_3, y_3\}$. If $s_1 x_3, s_1 y_3 \in E(G)$ then $G[s_1, u, x_3, y_3]$ contains K_4^-, so G contains TK_5 by Theorem 1.1. Thus we may assume $s_1 x_3 \notin E(G)$ or $s_1 y_3 \notin E(G)$. This implies $|V(R^*)| \geq 2$ (as s_1 has degree at least 5 in G). If $|V(R^*)| \geq 3$ then $|V(R_1)| \geq 7$; so by the symmetry between R_1 and H_1, we may assume $|V(H_1)| \geq 7$. Thus, we may assume $R^* = \{s_1, v\}$. Then v is adjacent to all of $\{b_0, b_1, s_1, x_3, y_3\}$. Since $s_1 x_3 \notin E(G)$ or $s_1 y_3 \notin E(G)$, $s_1 b_0, s_1 b_1 \in E(G)$; so $G[\{s_1, v, x_3, y_3\}]$ contains K_4^-; if $s_1 b_0, s_1 b_1 \in E(G)$ then $G[b_0, b_1, s_1, v]$ contains K_4^-. Hence G contains TK_5 by Theorem 1.1, completing the proof of (7).

We may assume by symmetry that S_0, S_1, S_2, S_3 end at x_1, y_1, y_2, x_2, respectively. If $H_1 - s_3$ contains no disjoint paths from x_3 to y_3 and from s_1 to s_2 then by Corollary 2.3, $(H_1 - s_3, x_3, s_1, y_3, s_2)$ is planar, and G contains TK_5 by (7) and Corollary 2.9. So we may assume that such disjoint paths exist in $H_1 - s_3$. These disjoint paths, $(K - x_2 y_3) \cup y_1 x_3 y_2 \cup b_0 S_0 x_1 \cup b_1 S_1 y_1 \cup S_2$, and three independent paths in $G[R^* + x_3]$ from s_1 to x_3, b_0, b_1, respectively (by (6)) form a TK_5 in G with branch vertices s_1, x_1, x_3, y_1, y_2. ∎

The next result will allow us to modify an existing x_1-x_2 path in H. An *endblock* in a graph K is a block in K containing at most one cutvertex of K.

Lemma 3.2 *Let Q be an x_1-x_2 path in H and let $B(Q)$ be a connected subgraph of $H - Q$. Then G contains TK_5, or H has an induced x_1-x_2 path Q' such that $H - Q'$ is connected and $B(Q) \subseteq H - Q'$, or H has an induced x_1-x_2 path Q' such that $H - Q'$ is connected and $\{y_1, y_2, y_3\} \in N(B(Q'))$ for some 2-connected block $B(Q')$ of $H - Q'$.*

Proof. Suppose for any induced x_1-x_2 path Z in H with $B(Q) \subseteq H - Z$, $H - Z$ has at least two components. Let $\beta(Z)$ denote the number of components in $H - Z$. We choose Z so that

(1) $\beta(Z)$ is minimum.

Let C denote a component of $H - Z$ such that $B(Q) \cap C = \emptyset$. Let $u_1, u_2 \in N(C) \cap V(Z)$ such that $u_1 Z u_2$ is maximal, and assume that x_1, u_1, u_2, x_2 occur on Z in order. Then

(2) $N(C \cup (u_1 Z u_2 - \{u_1, u_2\})) = \{u_1, u_2, y_1, y_2, y_3\}$.

For, otherwise, since G is 5-connected, $u_1 Z u_2 - \{u_1, u_2\}$ contains a neighbor of some component of $H - Z$ other than C.

We now use Lemma 2.1 to find a path P in $G[C + \{u_1, u_2\}]$ from u_1 to u_2. Let $B_1 \ldots B_k$ denote the chain of blocks in $G[C + \{u_1, u_2\}]$ from u_1 to u_2, with $u_1 \in B_1$ and $u_2 \in B_k$. Let C' be obtained from $G[C \cup u_1 Z u_2]$ by contracting $G[C \cup u_1 Z u_2] - \bigcup_{i=1}^{k} B_i$ to a single vertex u. Then $C' + u_1 u_2$ is 3-connected. So by Lemma 2.1, $C' + u_1 u_2$ contains an induced cycle T such that $u_1 u_2 \in E(T)$, $u \notin V(T)$ and $C' - T$ is connected. Let $P := T - u_1 u_2$.

Then $G[C \cup u_1 Z u_2] - P$ is connected. Let $Q' := u_1 Z x_1 \cup P \cup u_2 Z x_2$. Then Q' is an induced x_1-x_2 path in H. Since $(u_1 Z u_2 - \{u_1, u_2\}) \cap P = \emptyset$ and $u_1 Z u_2 - \{u_1, u_2\}$ contains a neighbor of some component of $H - Z$ other than C, we have $\beta(Q') < \beta(X)$, contradicting (1).

We may assume that

(3) $H - Z$ has just two components, namely C and the component D containing $B(Q)$, and if $w_1, w_2 \in N(D) \cap V(Z)$ such that $N(D) \cap V(Z) \subseteq V(w_1 Z w_2)$ then $u_1 Z u_2 \subseteq w_1 Z w_2$ and $\{u_1, u_2\} \neq \{w_1, w_2\}$.

Let D be an arbitrary component of $H - Z$ with $D \neq C$, and let $w_1, w_2 \in N(D) \cap V(Z)$ with $w_1 Z w_2$ maximal.

First, suppose $D \cap B(Q) = \emptyset$. By (2), we may use Menger's theorem to find in $G[C \cup u_1 Z u_2 + \{y_1, y_2, y_3\}]$ independent paths P_1, P_2, P_3, P_4, P_5 from some $x \in V(C)$ to u_1, u_2, y_1, y_2, y_3, respectively. If $u_1 Z u_2 \subseteq w_1 Z w_2$ or $w_1 Z w_2 \subseteq u_1 Z u_2$ then by (2), $\{u_1, u_2\} = \{w_1, w_2\} = N(D) \cap V(Z) = N(C) \cap V(Z)$; hence, since G is 5-connected, $y_1, y_2, y_3 \in N(D)$. In $G[D + \{y_1, y_2\}]$ we find a path P between y_1 and y_2. Now $(P_1 \cup u_1 Z x_1) \cup (P_2 \cup u_2 Z x_2) \cup P_3 \cup P_4 \cup P \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_1, y_2. Thus we may assume that $u_1 Z u_2 \not\subseteq w_1 Z w_2$ and $w_1 Z w_2 \not\subseteq u_1 Z u_2$. Then by (2) and by symmetry we may assume that $x_1, w_1, w_2, u_1, u_2, x_2$ occur on Z in this order. If $G[D \cup w_1 Z w_2 + \{y_1, y_2\}]$ contains disjoint paths Q_1, Q_2 from y_1, w_1 to y_2, w_2, respectively, then $(P_1 \cup u_1 Z w_2 \cup Q_2 \cup w_1 Z x_1) \cup (P_2 \cup u_2 Z x_2) \cup P_3 \cup P_4 \cup Q_1 \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_1, y_2. So assume that such Q_1, Q_2 do not exist. Then by (2) and by Corollary 2.3, $(G[D \cup w_1 Z w_2 + \{y_1, y_2\}], y_1, w_1, y_2, w_2)$ is planar. By Lemma 3.1, we may assume $|V(D) \cup V(w_1 Z w_2 - \{w_1, w_2\})| \geq 2$. So it follows from Corollary 2.9 that G contains TK_5.

Therefore, we may assume that $H - Z$ has only two components, namely C and D, and $B(Q) \subseteq D$. If $\{w_1, w_2\} = \{u_1, u_2\}$ then by (2), $N(D) \cap V(Z) = \{w_1, w_2\}$; so the argument in the first half of the above paragraph shows that G contains TK_5. Now suppose $u_1 Z u_2 \not\subseteq w_1 Z w_2$. Then by (2), we may assume that x_1, w_1, w_2, u_1, u_2 occur on Z in order. The argument in the second half of the above paragraph shows that G contains TK_5, completing the proof of (3).

By (2) and (3), we may assume that $x_1, w_1, u_1, u_2, w_2, x_2$ occur on Z in this order. Note by (2) that $\{u_1, u_2, y_1, y_2, y_3\}$ is a cut in G separating $C \cup u_1 Z u_2$ from D. By (3) and by symmetry, we may assume that $u_1 \neq w_1$. We now apply Lemma 2.1 as in the proof of (2) to find an induced w_1-w_2 path P in $G[D + \{w_1, w_2\}]$ such that $G[D \cup w_1 X w_2] - P$ is connected. Now let Q' be obtained from Z by replacing $w_1 Z w_2$ with P. Clearly, Q' is induced, and $H - Q'$ is connected. If $G[C \cup (u_1 Z u_2 - u_2)]$ is 2-connected, then it is contained in a 2-connected block of $H - Q'$, which is the desired $B(Q')$. So suppose $G[C \cup (u_1 Z u_2 - u_2)]$ is not 2-connected. By Lemma 3.1, we may assume that every vertex in $u_1 Z u_2 - \{u_1, u_2\}$ has at east two neighbors in C. So $G[C \cup (u_1 Z u_2 - u_2)]$ has an endblock, say C', disjoint from $u_1 Z u_2 - u_2$. Let v be the cut vertex of $G[C \cup (u_1 Z u_2 - u_2)]$ contained in C'. Since G is 5-connected, $y_1, y_2, y_3 \in N(C')$. By Lemma 3.1, we may assume that C' is 2-connected. So C' is contained in a 2-connected block of $H - Q'$, which is the desired $B(Q')$. ∎

The next lemma says that if H has an induced x_1-x_2 path Q such that $H - Q$ is connected then we can choose an x_1-x_2 path X so that the minimum degree of $H - X$ is at least 2. In particular, $H - X$ has a 2-connected block.

Lemma 3.3 *Let Q be an induced x_1-x_2 path in H such that $H - Q$ is connected. Then G contains TK_5, or H contains an induced x_1-x_2 path X such that $H - X$ is connected, contains all 2-connected blocks of $H - Q$, and has minimum degree at least 2.*

Proof. For an arbitrary induced x_1-x_2 path Z in H for which $H - Z$ is connected and contains all 2-connected blocks of $H - Q$, let $\alpha_1(Z)$ denote the number of vertices of $H - Z$ with degree at most 1 in $H - Z$, and let $\alpha_2(Z)$ denote the number of vertices of $H - Z$ with degree at least 2 in $H - Z$. (Note that such Z exists because of Q.) We choose such Z that $\alpha_1(Z)$ is minimum and, subject to this, $\alpha_2(Z)$ is maximum.

If $\alpha_1(Z) = 0$, then $X := Z$ is the desired path. So assume $\alpha_1(Z) \geq 1$, and let u be a vertex of degree 1 in $H - Z$.

Since G is 5-connected, it follows from Lemma 3.1 that we may assume that u has at least three neighbors on Z. Let $u_1, u_2 \in N(u) \cap V(Z)$ with $u_1 Z u_2$ maximal, and we may assume that x_1, u_1, u_2, x_2 occur on Z in order. Let $X = x_1 Z u_1 u u_2 Z x_2$. Clearly, X is an induced path in G, and all 2-connected blocks of $H - Z$ (hence those of $H - Q$) are contained in $H - X$.

By Lemma 3.1, we may assume that each vertex of $u_1 Z u_2 - \{u_1, u_2\}$ has at least 1 neighbor in $H - Z - u$. If $|V(u_1 Z u_2)| = 3$ then $G[u_1 Z u_2 + u] \cong K_4^-$; G contains TK_5 by Theorem 1.1. So assume $|V(u_1 Z u_2)| \geq 4$. Then $\alpha_1(X) \leq \alpha_1(Z)$ and $\alpha_2(X) > \alpha_2(Z)$, a contradiction. ∎

Recall that we wish to find an induced path X in H from x_1 to x_2 such that $H - X$ is 2-connected, which will be the work of the next two sections. But first we show that we can find a 3-planar chain C in H from x_1 to x_2 such that $H - C$ is 2-connected, and we also need $H - C$ to have neighbors of as many y_i as possible. This leads to the following notation:

$$\gamma(X) := \max\{|N(B) \cap \{y_1, y_2, y_3\}| : B \text{ is a 2-connected block of } H - X\},$$

and let $B(X)$ denote a 2-connected block of $H-X$ with $|N(B(X))\cap\{y_1,y_2,y_3\}|=\gamma(X)$.

By Lemmas 3.2 and 3.3, we may assume that there exists an induced x_1-x_2 path X in H such that $H-X$ is connected and has a 2-connected block. So $\gamma(X)$ and $B(X)$ are defined for such X. Throughout the rest of this paper, we choose X and $B(X)$ so that the following are satisfied in the order listed:

(1) $\gamma(X)$ is maximum,

(2) $|\{y_i : |N(y_i)\cap V(B(X))|\geq 2\}, 1\leq i\leq 3\}|$ is maximum, and

(3) $B(X)$ is maximal.

When understood, we will simply refer to $B(X)$ as B.

The following lemma allows us to exclude certain 2-cuts in H.

Lemma 3.4 *G contains TK_5, or H contains no 2-cut separating B from some vertex.*

Proof. Suppose that $\{u,v\}$ is a 2-cut in H separating B from some vertex. Let D denote a $\{u,v\}$-bridge of H not containing B. Since H is 2-connected, $\{u,v\}\subseteq V(D)$.

Suppose $u,v\in V(X)$. Then, since $H-X$ is connected and X is induced, $D=uXv$. Since G is 5-connected, each vertex of $D-\{u,v\}$ is adjacent to all of y_1,y_2,y_3. So by Lemma 3.1, G contains TK_5.

Thus we may assume that $\{u,v\}\not\subseteq V(X)$. Since $H-X$ is connected and B is a 2-connected block of $H-X$ (so $\{u,v\}\not\subseteq V(B)$), we may assume that H has disjoint paths P_u,P_v from u,v to $x\in V(X), b\in V(B)$, respectively, and internally disjoint from $B\cup D\cup X$, and that $v\notin X$. Then $u\notin B$ as B is a block in $H-X$. Since G is 5-connected, $\{y_1,y_2,y_3\}\subseteq N(D-\{u,v\})$.

We claim that $\{y_1,y_2,y_3\}\subseteq N(B)$. If $D-u$ is 2-connected then this follows from Lemma 3.2 and the choice of X (as $D-u\subseteq H-X$). So we may assume that $D-u$ is not 2-connected, and let C denote an endblock of $D-u$. Since G is 5-connected, $\{y_1,y_2,y_3\}\subseteq N(C)$. By Lemma 3.1, we may assume that C is 2-connected. Hence, since $C\subseteq H-X$, it follows from Lemma 3.2 and the choice of X that $\{y_1,y_2,y_3\}\subseteq N(B)$.

By Lemma 3.1, we may assume that no two of $\{y_1,y_2,y_3\}$ share a common neighbor. Thus, since B is 2-connected, $G[B+\{y_1,y_2,y_3\}]$ has two disjoint paths Q_1,Q_2 with ends in $\{b,y_1,y_2,y_3\}$. Without loss of generality, we may assume that Q_1 is between y_1 and y_2 and Q_2 is between y_3 and b.

If $G[D+\{y_1,y_2,y_3\}]-u$ contains disjoint paths R_1,R_2 from y_1,y_2 to v,y_3, respectively, then $Q_1\cup(Q_2\cup P_v\cup R_1)\cup R_2\cup X\cup K$ is a TK_5 in G with branch vertices x_1,x_2,y_1,y_2,y_3. So we may assume that such R_1,R_2 do not exist. Then by Corollary 2.3, $(G[D+\{y_1,y_2,y_3\}]-u,y_1,y_2,v,y_3)$ is planar. By Lemma 3.1 we may assume that $|V(D)-\{u,v\}|\geq 3$. Hence G contains TK_5 by Corollary 2.9. ∎

In [3], it is shown that 4-connected graphs contain non-separating planar chains between any two specific vertices. We now use a similar argument to show that $H-B$ is a 3-planar chain. We proceed by proving four lemmas.

Lemma 3.5 *Suppose H has two connected subgraphs C,D such that*

- $|V(C\cap B)|\leq 1$, $V(C\cap X)=\{u,v\}$, $C-(B\cup X)$ *is connected,*

- $|V(D \cap B)| \leq 1$, $V(D \cap X) = \{u, v\}$ *or* $V(D \cap X) = V(uXv)$, *and* $D - (B \cup X)$ *is connected, and*

- $\{u, v\} \cup V(C \cap B)$ *is a cut in* H *separating* C *from* $B \cup D \cup X$, *and* $\{u, v\} \cup V(D \cap B)$ *is a cut in* H *separating* D *from* $B \cup C \cup (X - uXv)$.

Then G *contains* TK_5.

Proof. Without loss of generality assume that x_1, u, v, x_2 occur on X in order. Let

$$S_C := \{u, v\} \cup V(C \cap B) \cup (N(C - \{u, v\} - V(C \cap B)) \cap \{y_1, y_2, y_3\})$$

and

$$S_D := \{u, v\} \cup V(D \cap B) \cup (N(D - \{u, v\} - V(D \cap B)) \cap \{y_1, y_2, y_3\}).$$

Since G is 5-connected, $|S_C| \geq 5$ and $|S_D| \geq 5$. By Lemma 3.4, we may assume $V(C \cap B) = \{c\}$ and $V(D \cap B) = \{d\}$.

We claim that $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$. Let A denote an endblock of $C - \{u, v\}$ and let $a \in V(A)$ such that if $A = C - \{u, v\}$ then $a = c$, and if $A \neq C - \{u, v\}$ then a is the cut vertex of $C - \{u, v\}$ contained in A. Since G is 5-connected, we see that $|N(A - a) \cap \{y_1, y_2, y_3\}| \geq 2$. By Lemma 3.1, we may assume that A is 2-connected. Hence the claim follows from the choice of X and Lemma 3.2.

By Lemma 2.4, $G[C + S_C]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $w \in V(C)$ to S_C such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 5$, $|V(P_i) \cap S_C| = 1$ for $1 \leq i \leq 5$, $u \in V(P_1)$, and $v \in V(P_2)$. By symmetry, we may assume that $y_1 \in V(P_3)$ and $y_2 \in V(P_4)$.

If $y_1, y_2 \in S_D$ then $G[D + \{y_1, y_2\}] - \{u, v\}$ contains a path Q between y_1 and y_2; and $(P_1 \cup uXx_1) \cup (P_2 \cup vXx_2) \cup P_3 \cup P_4 \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_2. Similarly, if $y_1, y_2 \in N(B)$ then $G[B + \{y_1, y_2\}]$ contains a path Q between y_1 and y_2; again $(P_1 \cup uXx_1) \cup (P_2 \cup vXx_2) \cup P_3 \cup P_4 \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_2.

Thus we may assume that $y_1 \notin S_D$ and $\{y_1, y_2\} \not\subseteq N(B)$. Hence $y_2, y_3 \in S_D$. By Menger's theorem, $G[D \cup S_D]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $x \in V(D)$ to u, v, y_2, y_3, d, respectively.

If $y_2, y_3 \in N(B)$ then $G[B + \{y_2, y_3\}]$ contains a path R between y_2 and y_3; so $(Q_1 \cup uXx_1) \cup (Q_2 \cup vXx_2) \cup Q_3 \cup Q_4 \cup R \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_2, y_3. Similarly, if $y_2, y_3 \in S_C$ then $G[C + \{y_2, y_3\}] - \{u, v\}$ has a path R between y_2 and y_3; again $(Q_1 \cup uXx_1) \cup (Q_2 \cup vXx_2) \cup Q_3 \cup Q_4 \cup R \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_2, y_3.

Hence we may assume that $\{y_2, y_3\} \not\subseteq N(B)$ and $\{y_2, y_3\} \not\subseteq S_C$. Therefore, $y_1, y_3 \in N(B)$ (since $y_1, y_2 \notin N(B)$) and $y_3 \notin S_C$ (since $y_2 \in S_C$). Thus $G[B + \{y_1, y_3\}]$ contains a path R_{13} between y_1 and y_3, and $G[C + \{y_1, y_2\}] - \{u, v, c\}$ contains a path R_{12} between y_1 and y_2. If $G[D + \{y_2, y_3\}] - d$ contains disjoint paths R_1, R_2 from u, y_2 to v, y_3, respectively, then $R_{12} \cup R_{13} \cup R_2 \cup (x_1Xu \cup R_1 \cup vXx_2) \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. So we may assume that R_1, R_2 do not exist. Then by Corollary 2.3, $(G[D + \{y_2, y_3\}] - d, u, y_2, v, y_3)$ is planar. Since $y_2, y_3 \in N(D - \{d, u, v\})$, we may assume by Lemma 3.1 that $|V(D) - \{d, u, v\}| \geq 2$. So G contains TK_5 by Corollary 2.9. ∎

Let $\mathcal{B}$ denote the set of B-bridges of $H - X$. For each $D \in \mathcal{B}$, $V(B) \cap V(D)$ consists of exactly one vertex, denoted by r_D. For any $x, y \in V(X)$, we denote $x \leq y$ if $x \in V(X[x_1, y])$.

If $x \le y$ and $x \neq y$, then we write $x < y$. By Lemma 3.4, we may assume that, for each $D \in \mathcal{B}$, $D - r_D$ has at least two neighbors on X. Let l_D and h_D denote the the neighbors of $D - r_D$ on X such that $l_D < h_D$ and $l_D X h_D$ is maximal. For each vertex u of $H - X$, we define $u^* = r_D$ if $u \in V(D)$ for some $D \in \mathcal{B}$, and $u^* = u$ if $u \in V(B)$. We say that a member D of $\mathcal{B}$ is a *nice bridge* if there exist $u, v \in N_H(l_D X h_D - \{l_D, h_D))$ such that $u, v \notin V(D - r_D) \cup V(X)$ and $u^* \neq v^*$.

Lemma 3.6 *G contains TK_5, or there is no nice B-bridge in H.*

Proof. Suppose D is a nice bridge in H. Let $u, v \in N_H(l_D X h_D - \{l_D, h_D\})$ such that $u, v \notin V(D - r_D) \cup V(X))$ and $u^* \neq v^*$. We now use Lemma 2.1 to find a path P in $G[D + \{l_D, h_D\}] - r_D$ from l_D to h_D.

Let $B_1 \dots B_k$ denote the chain of blocks in $G[D + \{l_D, h_D\}] - r_D$ from l_D to h_D, with $l_D \in V(B_1)$ and $h_D \in V(B_k)$. Let D' be obtained from $G[D \cup l_D X h_D]$ by identifying $G[D \cup l_D X h_D] - \bigcup_{i=1}^{k} B_i$ to a single vertex u. Then by Lemma 3.4, we may assume that $D' + u_1 u_2$ is 3-connected. So by Lemma 2.1, $D' + u_1 u_2$ contains an induced cycle T such that $u_1 u_2 \in E(T)$, $u \notin V(T)$, and $D' - T$ is connected. Let $P := T - u_1 u_2$. Then $G[D \cup l_D X h_D] - P$ has at most two components, each containing r_D or $l_D X h_D - \{l_D, h_D\}$.

Let $Q' := x_1 X l_D \cup P \cup h_D X x_2$. Then Q' is an induced x_1-x_2 path in H and $H - X'$ is connected. However, $H - X'$ has a block containing $B(X)$ and a path from u^* to v^* internally disjoint from $B(X)$, contradicting the choice of X. ∎

We say that two B-bridges C and D in $\mathcal{B}$ *overlap* if $E(l_C X h_C) \cap E(l_D X h_D) \neq \emptyset$. Define an auxiliary graph $\mathcal{G}$ with $V(\mathcal{G}) = \mathcal{B}$ such that $C, D \in \mathcal{B}$ are adjacent in $\mathcal{G}$ if, and only if, C and D overlap. The following two lemmas are similar to results in [2,3]. The difference is that we need Lemma 3.5 here instead of 4-connectedness in [2,3].

Lemma 3.7 *Let $D_1 D_2 D_3$ be a path in $\mathcal{G}$. Then $|\{r_{D_i} : i = 1, 2, 3\}| \le 2$ or G contains TK_5. Moreover, if $D_1 D_2 D_3$ is an induced path in $\mathcal{G}$ then $r_{D_1} = r_{D_3}$ or G contains TK_5.*

Proof. First, suppose $D_1 D_2 D_3$ is an induced path in $\mathcal{G}$. Then D_1 and D_3 do not overlap. Thus we may assume $l_{D_1} < h_{D_1} \le l_{D_3} < h_{D_3}$. Moreover, $l_{D_2} < h_{D_1}$ and $l_{D_3} < h_{D_2}$. Let $u \in V(D_1) - \{r_{D_1}\}$ such that $u h_{D_1} \in E(G)$ and let $v \in V(D_3) - \{r_{D_3}\}$ such that $v l_{D_3} \in E(G)$. Clearly, $u, v \in N_H(l_{D_2} X h_{D_2} - \{l_{D_2}, h_{D_2}\})$, $u, v \notin (V(D_2) - \{r_{D_2}\}) \cup V(X)$, and $u^* = r_{D_1}$ and $v^* = r_{D_3}$. So by Lemma 3.6, $r_{D_1} = r_{D_3}$ or G contains TK_5.

Now assume that D_1 and D_3 overlap. By symmetry, we may assume that $l_{D_1} X h_{D_1}$ is not properly contained in $l_{D_i} X h_{D_i}$ for $i = 2, 3$. Then for each $i \in \{2, 3\}$, either $l_{D_i} X h_{D_i} = l_{D_1} X h_{D_1}$, or $l_{D_i} \in V(l_{D_1} X h_{D_1}) - \{l_{D_1}, h_{D_1}\}$, or $h_{D_i} \in V(l_{D_1} X h_{D_1}) - \{l_{D_1}, h_{D_1}\}$. Therefore, by Lemma 3.5 and by relabeling D_1, D_2, D_3 (if necessary), we may assume that there exist $x \in V(l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}) \cap N(D_2 - r_{D_2})$ and $y \in V(l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}) \cap N(D_3 - r_{D_3})$. Let u be a neighbor of x in $D_2 - r_{D_2}$, and v be a neighbor of y in $D_3 - r_{D_3}$. Then $u^* = r_{D_2}$ and $v^* = r_{D_3}$. By Lemma 3.6, we may assume $u^* = v^*$; so $|\{r_{D_i} : i = 1, 2, 3\}| \le 2$. ∎

Lemma 3.8 *Let $\mathcal{G}_i$, $i = 1, \dots, k$, denote the components of the graph $\mathcal{G}$. Then $|\{r_D : D \in V(\mathcal{G}_i)\}| \le 2$ for all $i = 1, \dots, k$, or G contains TK_5.*

Proof. For, suppose $|\{r_D : D \in V(\mathcal{G}_i)\}| \geq 3$ for some $1 \leq i \leq k$. Choose $D_1, D_2, D_3 \in V(\mathcal{G}_i)$ such that $r_{D_1}, r_{D_2}, r_{D_3}$ are pairwise distinct and, subject to this, the connected subgraph of $\mathcal{G}_i$ containing $\{D_1, D_2, D_3\}$, denoted by $\mathcal{T}$, has the minimum number of edges.

Thus, $\mathcal{T}$ is a tree whose leaves belong to $\{D_1, D_2, D_3\}$. So we may assume that D_1 and D_2 are two leaves of $\mathcal{T}$. Then $r_{D_j} \neq r_D$ for $j = 1, 2$ and for all $D \in V(\mathcal{T}) - \{D_j\}$; as otherwise $\mathcal{T} - D_j$ contradicts the minimality of $\mathcal{T}$. We may assume that $|\mathcal{T}| \geq 4$; otherwise, G contains TK_5 by Lemma 3.7. Thus, D_3 is not a leaf of $\mathcal{T}$; otherwise, $\mathcal{T} - D_3$ contradicts the minimality of $\mathcal{T}$. Therefore, $\mathcal{T}$ is actually a path between D_1 and D_2. Hence, since $|\mathcal{T}| \geq 4$ and $|\mathcal{T}|$ is minimum, $\mathcal{T}$ has a subpath of length 2 with ends D_1 and D such that this path is induced in $\mathcal{G}$ and $r_{D_1} \neq r_D$; so G contains TK_5 by Lemma 3.7. ∎

We are now ready to assume that $H - B$ is a 3-planar chain.

Lemma 3.9 *G contains TK_5, or $H - B$ is a 3-planar chain from x_1 to x_2.*

Proof. Let $\mathcal{G}_i$, $i = 1, \ldots, k$, denote the components of the graph $\mathcal{G}$. For each i, $\bigcup_{D \in V(\mathcal{G}_i)} l_D X h_D$ is a subpath of X; and let $u_i \leq v_i$ denote the ends of this path. By Lemma 3.4, we may assume $u_i < v_i$ for all i. Let B_i denote the subgraph of $H - B$ that is induced by the union of $u_i X v_i$ and $D - r_D$ for all $D \in V(\mathcal{G}_i)$. Then $B_i \cap X_i$, $i = 1, \ldots, k$, are pairwise edge-disjoint, and no cut vertex of B_i separates u_i from v_i. By Lemma 3.8, $|N(B_i - \{u_i, v_i\}) \cap V(B)| \leq 2$.

Suppose $|V(B_i)| \geq 3$. Then, since B is a block of $H - X$, we may assume by Lemma 3.4 that B_i is 2-connected. Since X is induced and $H - X$ is connected, $|N(B_i - \{u_i, v_i\}) \cap V(B)| \geq 1$. If $|N(B_i - \{u_i, v_i\}) \cap V(B)| = 1$ then by Lemma 3.5, we may assume that $B_i - \{u_i, v_i\}$ is connected. Now assume $N(B_i - \{u_i, v_i\}) \cap V(B) = \{w_1, w_2\}$.

We may assume that $(G[B_i + \{w_1, w_2\}] - w_1w_2, u_i, w_1, v_i, w_2)$ is 3-planar. For, otherwise, it follows from Lemma 2.2 that $B'_i := G[B_i + \{w_1, w_2\}] - w_1w_2$ contains disjoint paths P, Q from u_i, w_1 to v_i, w_2, respectively. Let C denote a chain of blocks in $B'_i - Q$ from u_i to v_i. Since B_i is 2-connected, $B'_i - C$ is connected. Let C' be obtained from $B'_i + u_iv_i$ by contracting $B'_i - C$ to a single vertex u. Note that C' is 2-connected and $C' - u$ is 2-connected. Suppose C' is not 3-connected, and let T be a cut set in C' such that $u \notin T$ and $|T| = 2$. Since $u_iv_i \in E(C')$, T is also a cut in H separating B from some vertex. Therefore, G contains TK_5 by Lemma 3.4. So we may assume that C' is 3-connected. Then by Lemma 2.1, C' contains an induced path P' from u_i to v_i such that $u \notin V(P')$ and $C' - P'$ is connected. Let X'' be obtained from X by replacing $u_i X v_i$ by P'. Then $H - X''$ is connected, and $B \cup Q$ is contained in a 2-connected block of $H - X''$, contradicting the maximality of B.

We may assume that $B_i - \{u_i, v_i\}$ is connected. For suppose not, and let C_1, C_2 denote two components of $B_i - \{u_i, v_i\}$. Since B_i is 2-connected, $\{u_i, v_i\} \subseteq N(C_j)$ for $j = 1, 2$. So by the above claim we may assume that $w_1 \notin N(C_2)$ and $w_2 \notin N(C_1)$. Now by Lemma 3.5, G contains TK_5. Therefore, $H - B$ is a 3-planar chain. ∎

We adopt the following notation throughout the rest of this paper. Let D be a block in $H - B$, and let $u_D, v_D \in V(D \cap X)$ with $u_D X v_D$ maximal such that x_1, u_D, v_D, x_2 occur on X in order. If $|N(D - \{u_D, v_D\}) \cap V(B)| = 2$, let $N(D - \{u_D, v_D\}) \cap V(B) = \{b_D, c_D\}$, and we say that D is a block of *type I* in $H - B$. If $|N(D - \{u_D, v_D\}) \cap V(B)| = 1$, let $N(D - \{u_D, v_D\}) \cap V(B) = \{b_D\}$ and $c_D = b_D$, and call D a block of *type II* in $H - B$. Also, let D' be obtained from $G[D + \{b_D, c_D\}]$ by deleting edges from $\{b_D, c_D\}$ to $\{u_D, v_D\}$. Note that $D' - \{b_D, c_D\} = D$ which is 2-connected when $|V(D)| \geq 3$.

4 Blocks of type I

The aim of this section is to show that if there is a block of type I in $H - B$, then G contains TK_5. So let D be a block of type I in $H - B$, and recall the notation for D', b_D, c_D, u_D, v_D. Also recall that D' contains no edge from $\{b_D, c_D\}$ to $\{u_D, v_D\}$, $b_D, c_D \in B$, and x_1, u_D, v_D, x_2 occur on X in order.

We will be interested in the graph obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 as y. The idea is to apply Corollaries 2.11 and 2.12 to this graph; so we need it to be $(5, \{b_D, c_D, u_D, v_D, y\})$-connected. Thus, we need to know when D' is not $(4, \{b_D, c_D, u_D, v_D\})$-connected.

Lemma 4.1 *Suppose S is a minimal cut in D' such that $|S| \leq 3$ and $D' - S$ has a component C disjoint from $\{b_D, c_D, u_D, v_D\}$. Then G contains TK_5, or $|S| = 3$ and one of the following holds:*

(i) $D - C$ contains a path P from u_D to v_D such that $S \not\subseteq V(P)$, or

(ii) $S \cap \{b_D, c_D, u_D, v_D\} = \{v_D\}$, and $S - \{v_D\}$ is a 2-cut in D' separating $C + v_D$ from $\{b_D, c_D, u_D\}$, or

(iii) $S \cap \{b_D, c_D, u_D, v_D\} = \{u_D\}$, and $S - \{u_D\}$ is a 2-cut in D' separating $C + u_D$ from $\{b_D, c_D, v_D\}$.

Proof. By Lemma 3.4, we may assume $|S| = 3$. Note that $S \subseteq N(C)$ by the minimality of S. Suppose $D - C$ contains no path from u_D to v_D. Then let C_1, C_2 denote the components of $D - C$ containing u_D, v_D, respectively. Since $|S| = 3$, $|S \cap V(C_1)| \leq 1$ or $S \cap V(C_2)| \leq 1$. Suppose $|S \cap V(C_2)| \leq 1$. Because D is 2-connected, we must have $S \cap V(C_2) = \{v_D\}$ and $b_D, c_D \notin S$. Note that b_D, c_D have no neighbors in C and, in D', neither b_D nor c_D is adjacent to v_D. So $S - \{v_D\}$ is a 2-cut in D' separating $C + v_D$ from $\{b_D, c_D, u_D\}$, and (ii) holds. Similarly, if $|S \cap V(C_1)| \leq 1$ then (iii) holds.

Thus we may assume that $D - C$ contains a path P from u_D to v_D. If $S \not\subseteq V(P)$, then (i) holds. So we may assume that $S \subseteq V(P)$ for any path P in $D - C$ from u_D to v_D.

Let $s_1, s_2 \in S$ with $s_1 P s_2$ maximal, and assume that u_D, s_1, s_2, v_D occur on P in order. Since (D', b_D, u_D, c_D, v_D) is 3-planar, D' is the union of two subgraphs D_1 and D_2 such that $V(D_1 \cap D_2) \subseteq V(P)$, $E(D_1) \cap E(D_2) = \emptyset$, $b_D \in V(D_1)$ and $c_D \in V(D_2)$. Note that $s_2 = v_D$, or $\{s_2, c_D\}$ is a 2-cut in D_2 separating v_D from u_D; otherwise we can modify P inside D_2 to avoid s_2. Similarly, $s_2 = v_D$, or $\{b_D, s_2\}$ is a 2-cut in D_1 separating v_D from u_D. Since D is 2-connected, we must have $s_2 = v_D$. By the same argument, we also have $s_1 = u_D$. Since S is minimal and C is connected, $C \subseteq D_1$ or $C \subseteq D_2$. However, as (D', b_D, u_D, c_D, v_D) is 3-planar, $\{u_D, v_D\}$ must be a cut in D' separating b_D from c_D. Thus G contains TK_5 by Lemma 3.5. ∎

The next result will allow us to assume that D' is $(4, \{b_D, c_D, u_D, v_D\})$-connected.

Lemma 4.2 *Suppose S is a minimal cut in D' and C is a component of $D' - S$ such that $|S| \leq 3$ and $V(C) \cap \{b_D, c_D, u_D, v_D\} = \emptyset$. Then G contains TK_5.*

Proof. Note that the minimality of S implies $S \subseteq N(C)$. We choose S and C so that

(1) C is maximal and, subject to this, $|S \cap \{b_D, c_D\}|$ is minimum.

Since D is 2-connected, $|S - \{b_D, c_D\}| \geq 2$ and there exist $s, t \in S - \{b_D, c_D\}$ such that

(2) $D - (S - \{s, t\})$ contains disjoint paths P', P'' from s, t to u_D, v_D, respectively.

By Lemma 4.1, we may assume that $|S| = 3$, and (i) or (ii) or (iii) of Lemma 4.1 holds. Let $S - \{s, t\} = \{r\}$. Since G is 5-connected, $|N(C) \cap \{y_1, y_2, y_3\}| \geq 2$. We may assume that

(3) $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$ for any 2-connected endblock C' of C, and $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

First, suppose there is a path P in $D - C$ from u_D to v_D such that $S \not\subseteq V(P)$, and let X' be obtained from X by replacing $u_D X v_D$ with P. Then $C' \subseteq H - X'$; so by Lemma 3.2 and the choice of X, we have $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$ for any 2-connected block C' of C. If C is 2-connected, then $C' = C$ and hence $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}| \geq 2$; so (3) holds. Thus we may assume that C is not 2-connected. Let $C_1, \ldots, C_k$ denote the endblocks of C, where $k \geq 2$. Suppose $|N(C_i) \cap S| \leq 2$ for some i. Then, since G is 5-connected, $|N(C_i) \cap \{y_1, y_2, y_3\}| \geq 2$. Hence by Lemma 3.1, we may assume that C_i is 2-connected. So C_i is contained in a 2-connected block of $H - X'$, and we may assume (3) by the choice of X and Lemma 3.2. So we may assume $S \subseteq N(C_i)$ for $i = 1, \ldots, k$. This implies that $G[C + (S - V(P))]$ is 2-connected, and hence is contained in a 2-connected block of $H - X'$. By the choice of X and by Lemma 3.2, we may assume (3).

Now, suppose that there is no path in $D - C$ from u_D to v_D such that $S \not\subseteq V(P)$. Then by symmetry, we may further assume that S, C satisfy (ii) of Lemma 4.1. Then $v_D = t$, and $b_D, c_D \notin S$. Since G is 5-connected, $|N(C) \cap \{y_1, y_2, y_3\}| \geq 2$. So by Lemma 3.1, $|V(C)| \geq 3$.

We claim that $v_D = x_2$ and there is no path in H from x_2 to B internally disjoint from $B \cup X \cup C$. For, otherwise, $H - C$ contains a path X' between x_1 and x_2 (which could use a path in $D - C$ from b_D or c_D to u_D). So by Lemma 3.2 and the choice of X, we may assume $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$ for any 2-connected block C' of C. Clearly, $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$ if $|N(C') \cap \{y_1, y_2, y_3\}| \geq 2$ for some choice of C'. So assume $|N(C') \cap \{y_1, y_2, y_3\}| \leq 1$ for any choice of C'. Then $C' \neq C$ and $S \subseteq N(C')$ (since G is 5-connected); so $G[C + S] - v_D$ is 2-connected and contained in $H - X'$. So by Lemma 3.2 and the choice of X we may assume $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

Note that $S - \{v_D\}$ is a 2-cut in D separating $v_D = x_2$ from $\{b_D, c_D, u_D\}$. Let J denote the $(S - \{v_D\})$-bridge of D containing C and $v_D = x_2$. Suppose J is not 2-connected, and let z be a cut vertex of J. Since D is 2-connected, z must separate some $r \in S - \{v_D\}$ from $S - \{r\}$. By Lemma 3.4, we may assume that the z-bridge of J containing r is induced by the edge rz. Let J' be obtained from J by deleting every vertex in $S - \{v_D\}$ that has degree 1 in J; then J' is 2-connected. Let $T = \{v_1, v_2\} \subseteq V(J')$ be the cut of D separating J' from $\{b_D, c_D, u_D\}$. Since G is 5-connected and $|C| \geq 3$, we may assume $y_2, y_3 \in N(J' - \{v_1, v_2, x_2\})$. So by Lemma 3.1, we may assume $|V(J')| \geq 5$.

Note that $\{v_1, v_2, y_1, y_2, y_3\}$ is a cut in G, and we can write $G = G_1 \cup G_2$ such that $V(G_1 \cap G_2) = \{v_1, v_2, y_1, y_2, y_3\}$, $J' \subseteq G_1$, and $B \subseteq G_2$. Since $G_2 - \{v_1, v_2, x_1, y_1\}$ is connected, it contains three independent paths from some vertex $u \in V(G_2 - x_1) - V(G_1)$ to x_1, y_2, y_3, respectively. Thus by Lemma 2.4, G_2 has five independent paths P_1, P_2, P_3, P_4, P_5 from u to

$S' := \{v_1, v_2, x_1, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \le i < j \le 5$, $|V(P_i) \cap S'| = 1$ for $1 \le i \le 5$, $x_1 \in V(P_1)$, $y_2 \in V(P_2)$, and $y_3 \in V(P_3)$. We may assume that P_4 ends in $\{v_1, v_2\}$.

We may assume that $y_1 \in N(J' - \{v_1, v_2, x_2\})$. For, suppose not. Then $\{v_1, v_2, x_2, y_2, y_3\}$ is a 5-cut in G. Without loss of generality, assume $v_1 \in V(P_4)$. If $G[J' + \{y_2, y_3\}] - v_2$ contains disjoint paths Q_1, Q_2 from v_1, y_2 to x_2, y_3, respectively, then $P_1 \cup (P_4 \cup Q_1) \cup P_2 \cup P_3 \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume such Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[J' + \{y_2, y_3\}] - v_2, v_1, y_2, x_2, y_3)$ is planar. So G contains TK_5 by Corollary 2.9.

We claim that for any v_i, there exists $\{p, q\} \subseteq \{1, 2, 3\}$ such that $G[J' + \{y_p, y_q\}]$ contains disjoint paths from v_i, y_p to x_2, y_q, respectively. To prove this let J'' be obtained from $G[J' + \{y_1, y_2, y_3\}]$ by identifying y_1 and y_2 as y. If J'' contains disjoint paths from v_1, y to x_2, y_3, respectively, then this claim holds for some $p \in \{1, 2\}$ and $q = 3$. Otherwise, by Lemma 2.2, (J'', v_1, y, x_2, y_3) is planar. Then since J' is 2-connected, we see that the claim holds for $p = 1$ and $q = 2$.

Now we may assume without loss of generality that $G[J' + \{y_2, y_3\}]$ contains disjoint paths R_1, R_2 from v_1, y_2 to x_2, y_3, respectively. (The notation can be chosen this way so that we can use the paths $P_1, \ldots, P_5$ above.) If $v_1 \in V(P_k)$ for some $k \in \{4, 5\}$, then $P_1 \cup (P_k \cup R_1) \cup P_2 \cup P_3 \cup R_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume $v_1 \notin V(P_4 \cup P_5)$. Hence we may further assume that $v_2 \in V(P_4)$ and $y_1 \in V(P_5)$. Now by the above claim there exists $\{p, q\} \subseteq \{1, 2, 3\}$ such that $G[J' + \{y_p, y_q\}]$ contains disjoint paths R_1', R_2' from v_2, y_p to x_2, y_q, respectively. Then $P_1 \cup (P_4 \cup R_1') \cup R_2' \cup K$ and $P_2 \cup P_3$ (if $\{p, q\} = \{2, 3\}$), or $P_2 \cup P_5$ (if $\{p, q\} = \{1, 2\}$), or $P_3 \cup P_5$ (if $\{p, q\} = \{1, 3\}$) form a TK_5 in G with branch vertices u, x_1, x_2, y_p, y_q. This completes the proof of (3).

(4) We may assume $\{y_1, y_2, y_3\} \not\subseteq N(C)$.

Suppose $\{y_1, y_2, y_3\} \subseteq N(C)$. Let $S' := S \cup \{y_1, y_2, y_3\}$.

We may assume $\{y_1, y_2, y_3\} \not\subseteq N(B)$. For, suppose $\{y_1, y_2, y_3\} \subseteq N(B)$. Since $G[C + \{y_1, s, t\}]$ is connected, it contains three independent paths from some vertex $u \in V(C)$ to y_1, s, t, respectively. So Lemma 2.4 implies the existence of five independent paths P_1, P_2, P_3, P_4, P_5 in $G[C + S']$ from u to S', such that $V(P_i \cap P_j) = \{u\}$ for $1 \le i < j \le 5$, $|V(P_i) \cap S'| = 1$ for $1 \le i \le 5$, $y_1 \in V(P_1)$, $s \in V(P_3)$, and $t \in V(P_4)$. We may assume by symmetry (between y_2 and y_3) that P_2 ends at y_2, and let Q denote a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2. Then by (2), $(P_3 \cup P' \cup u_D X x_1) \cup (P_4 \cup P'' \cup v_D X x_2) \cup P_1 \cup P_2 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

If (i) of Lemma 4.1 holds, then let X' be the path obtained from X by replacing $u_D X v_D$ with P. We may assume that the paths P' and P'' in (2) are subpaths of P. Then $G[C + r] \subseteq H - X'$. If $G[C + r]$ is 2-connected then by (3), it follows from Lemma 3.2 and the choice of X that G contains TK_5 as $\{y_1, y_2, y_3\} \subseteq N(C)$ and $\{y_1, y_2, y_3\} \not\subseteq N(B)$. So we may assume $G[C + r]$ is not 2-connected. Let J be an endblock of $G[C + r]$ and v be the cutvertex of $G[C + r]$ contained in J such that $r \notin V(J - v)$. If $\{y_1, y_2, y_3\} \subseteq N(J - v)$ then by (3), it follows from Lemma 3.2 and the choice of X that G contains TK_5 as $\{y_1, y_2, y_3\} \not\subseteq N(B)$. Hence we may assume $y_1, y_2 \in N(J - v)$ and $y_3 \notin N(J - v)$; so $s, t \in N(J - v)$. By Menger's theorem, $G[J + \{s, t, y_1, y_2\}]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $u \in V(J - v)$ to y_1, y_2, s, t, v, respectively. Since $y_3 \in N(C)$ we see that P_5 can be extended through $G[C-(J-v)+y_3]$ to a path Q_5' ending at y_3. If $y_1, y_2 \in N(B)$ then let Q be a path in $G[B+\{y_1,y_2\}]$ between y_1 and y_2; now $Q_1 \cup Q_2 \cup (Q_3 \cup P' \cup u_D X x_1) \cup (Q_4 \cup P'' \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume by (3) that $y_i, y_3 \in N(B)$ for some $i \in \{1,2\}$. Let Q' be a path in $G[B+\{y_i,y_3\}]$ between y_i and y_3. Then $Q_i \cup Q_5' \cup (Q_3 \cup P' \cup u_D X x_1) \cup (Q_4 \cup P'' \cup v_D X x_2) \cup Q' \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_3.

Therefore, we may assume by symmetry that (ii) of Lemma 4.1 holds. So $t = v_D$. Without loss of generality and by (3), assume $y_1, y_2 \in N(B)$. Note that $G[C+\{t,y_1,y_2\}]$ contains independent paths from some $u \in V(C)$ to y_1, y_2, t, respectively. So by Lemma 2.4, $G[C+S']$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to S' such that $V(Q_i \cap Q_j) = \{u\}$ for $1 \le i < j \le 5$, $|V(Q_i) \cap S'| = 1$ for $1 \le i \le 5$, $y_1 \in V(Q_1)$, $y_2 \in V(Q_2)$, and $t \in V(Q_3)$. We may assume that Q_4 ends at $v \in \{r,s\}$. Since D is 2-connected, $D-C$ contains a path R from v to u_D. Let Q be a path in $G[B+\{y_1,y_2\}]$ between y_1 and y_2. Then $Q_1 \cup Q_2 \cup (Q_3 \cup v_D X x_2) \cup (Q_4 \cup R \cup u_D X x_1) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. This completes the proof of (4).

By (4), let $y_1, y_2 \in N(C)$ and $y_3 \notin N(C)$. Since G is 5-connected, $C' := G[C+(S \cup \{y_1,y_2\})]$ is $(5, S \cup \{y_1,y_2\})$-connected. By Menger's theorem, C' contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $z \in V(C)$ to y_1, y_2, s, t, r, respectively.

If $y_1, y_2 \in N(B)$, then $G[B+\{y_1,y_2\}]$ contains a path A from y_1 to y_2. So by (2), $P_1 \cup P_2 \cup (P_3 \cup P' \cup u_D X x_1) \cup (P_4 \cup P'' \cup v_D X x_2) \cup A \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z.

Hence we may assume that $y_1 \notin N(B)$. So by (3), $y_2, y_3 \in N(B)$. Let Q denote a path in $G[B+\{y_2,y_3\}]$ between y_2 and y_3.

(5) We may assume $y_3 \notin N(D-\{u_D,v_D\})$.

Suppose $y_3 \in N(D-\{u_D,v_D\})$. First, consider the case when $r \notin \{b_D, c_D\}$. If $G[D-C+y_3]$ contains disjoint paths Q_1, Q_2, Q_3 from S to u_D, v_D, y_3, respectively, then by symmetry among r, s and t, we may assume that $s \in Q_1$ and $t \in Q_2$; now $P_2 \cup (P_5 \cup Q_3) \cup (P_3 \cup Q_1 \cup u_D X x_1) \cup (P_4 \cup Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_2, y_3, z. So we may assume that $G[D-C+y_3]$ has a minimal cut T, $|T| \le 2$, separating S from $\{u_D, v_D, y_3\}$. Thus $T-\{y_3\}$ is a cut in D separating $C+S$ from $\{u_D,v_D\}$. Since D is 2-connected, $y_3 \notin T$ and $|T|=2$. Let D_1 denote the T-bridge of D containing C (so $D_1 - T$ is connected), and let D_2 denote the minimal union of T-bridges of D containing $\{u_D,v_D\}$ (so D_2 consists of at most two T-bridges of D). If neither b_D nor c_D has a neighbor in $D_1 - T$, then T is a cut of D' separating D_1 from $\{b_D, c_D, u_D, v_D\}$; so $T \cup \{y_1,y_2\}$ is a 4-cut in G, a contradiction. Hence, we may assume that b_D has a neighbor in $D_1 - T$. If c_D has no neighbor in $D_1 - T$ then $T \cup \{b_D\}$ is a minimal cut of D' separating D_1 from $\{b_D, c_D, u_D, v_D\}$; so $T \cup \{b_D\}, D_1$ contradict the choices of S, C in (1). Hence we may assume that c_D also has a neighbor in $D_1 - T$. Then $G[D_1 - T + \{b_D, c_D\}]$ contains a path from b_D to c_D. Since (D', b_D, u_D, c_D, v_D) is 3-planar, it follows from Lemma 2.2 that D' contains no disjoint paths from b_D to c_D and from u_D to v_D. Hence, u_D and v_D belong to different components of D_2, and this contradicts the 2-connectedness of D and completes the proof of (5).

Now consider the case when $r \in \{b_D, c_D\}$, say $r = b_D$. Let $w \in N(y_3) \cap V(D-\{b_D,c_D\})$.

We claim that $D-C-(\{s,t\}-\{w\})$ has independent paths W_1, W_2 from w to u_D, v_D, respectively. For otherwise, $D-C$ has a cut T' separating $\{u_D,v_D\}$ from w such that $\{s,t\}-$

$\{w\} \subseteq T'$, $|T'| \leq 3$, and $|T'| \leq 2$ when $w \in \{s,t\}$. Let D_1 denote the T'-bridge of $D - C$ containing w. Since D is 2-connected and (D', b_D, u_D, c_D, v_D) is 3-planar, c_D has no neighbor in $D_1 - T'$. Moreover, if b_D has a neighbor in $D_1 - T'$ then b_D and two vertices from T' form a 3-cut in D' separating $C + w$ from $\{b_D, c_D, u_D, v_D\}$, contradicting the choice of S in (1).

Suppose $w \in \{s,t\}$, say $w = t$. Then $(W_1 \cup u_D X x_1) \cup (W_2 \cup v_D X x_2) \cup wy_3 \cup (P_4 \cup P_2) \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_3, y_2. So assume $w \notin \{s,t\}$. By Lemma 2.4, $G[D' - C + \{y_1, y_2\}]$ has four independent paths Q_1, Q_2, Q_3, Q_4 from w to $S' := \{b_D, c_D, u_D, v_D, s, t, y_1, y_2\}$ such that $|V(Q_i \cap Q_j) = \{w\}$ for $1 \leq i < j \leq 4$, $|V(Q_i) \cap S'| = 1$ for $1 \leq i \leq 4$, $u_D \in V(Q_1)$, and $v_D \in V(Q_2)$.

If $t \in V(Q_3)$ then $(Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup wy_3 \cup (Q_3 \cup P_4 \cup P_2) \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_3, y_2. So we may assume $t \notin V(Q_3)$. Similarly, we may assume $t \notin V(Q_4)$, $s \notin V(Q_3)$ and $s \notin V(Q_4)$.

If $y_i \in V(Q_3)$ for some $i \in \{1,2\}$ then in $G[B \cup C]$ we find a path R from y_3 to y_i; now $(Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup wy_3 \cup Q_3 \cup R \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_3, y_i. Thus we may assume $y_i \notin V(Q_3)$ for $i \in \{1,2\}$. Similarly, we may assume that $y_i \notin V(Q_4)$ for $i \in \{1,2\}$.

Thus, we may assume $b_D \in V(Q_3)$ and $c_D \in V(Q_4)$. By symmetry we may assume that in D', $Q_3 \cup Q_4$ separates $C \cup Q_1$ from Q_2. Let (D_1, D_2) denote a separation of D' such that $V(D_1 \cap D_2) \subseteq V(Q_3 \cup Q_4)$, $C \cup Q_1 \subseteq D_1$ and $Q_2 \subseteq D_2$. Then, since D is 2-connected and (D', b_D, u_D, v_D, c_D) is 3-planar, $D_2 - b_D$ has independent paths Q_1', Q_3' from w to $u_D, \{s,t\}$, respectively. Now with Q_1', Q_3' in place of Q_1, Q_3 above, we can show that G has a TK_5, completing the proof of (5).

Let y_3' be a neighbor of y_3 in B. By symmetry, assume $r \neq c_D$. If there is a minimal cut T in $D' - c_D$, with $|T| \leq 2$, separating $C \cup S$ from $\{b_D, u_D, v_D\}$, then T or $T \cup \{c_D\}$ contradicts the choice of S in (1). So we may assume that

(6) $D' - C - c_D$ has disjoint paths R_1, R_2, R_3 from u_D, v_D, b_D to s, t, r, respectively.

(7) We may assume $N(y_3) \cap V(B) = \{y_3'\}$.

For, suppose there exists $y_3'' \in N(y_3) \cap V(B) - \{y_3'\}$. If $N(y_2) \cap V(B) \neq \{b_D\}$ then $G[B + \{y_2, y_3\}]$ has independent paths Q_1, Q_2 from y_3 to b_D and y_2, respectively; so $P_2 \cup (P_5 \cup R_3 \cup Q_1) \cup (P_3 \cup R_1 \cup u_D X x_1) \cup (P_4 \cup R_2 \cup v_D X x_2) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_2, y_3, z.

Thus, we may assume $N(y_2) \cap V(B) = \{b_D\}$. Then for some $i \in \{1,2\}$, G has a path R from x_i to $r \in V(B)$ and internally disjoint from $X \cup B \cup D'$; otherwise, $\{x_1, x_2\} = \{u_D, v_D\}$, and either $\{b_D, c_D, u_D = x_1, y_3\}$ or $\{b_D, c_D, v_D = x_2, y_3\}$ is a 4-cut in G separating B from D, a contradiction. By symmetry, assume $i = 1$. Then let R_1', R_2' be independent paths in $G[B \cup R + y_3]$ from b_D to r, y_3, respectively. Now $(P_5 \cup P_2 \cup y_2 b_D) \cup R_1' \cup (P_3 \cup R_1 \cup u_D X x_1) \cup (P_4 \cup R_2 \cup v_D X x_2) \cup (R_2' \cup y_3 x_2) \cup x_1 y_1 x_2$ is a TK_5 in G with branch vertices b_D, z, x_1, x_2, y_2. So we have (7).

(8) We may further assume that $H - B$ has a 2-connected block F such that $y_3 \in N(F)$, $y_3' \in \{b_F, c_F\}$, and $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order.

For, otherwise, by (7) and by symmetry, we may assume that y_3 has a neighbor $y_3'' \in V(u_D X x_1 - x_1)$. If $y_3 \in N(u_D)$ then we find independent paths L_1, L_2 in $G[D + y_2]$ from

u_D to y_2, v_D, respectively; now $u_D X x_1 \cup (L_2 \cup v_D X x_2) \cup L_1 \cup u_D y_3 \cup Q \cup K$ is a TK_5 in G with branch vertices u_D, x_1, x_2, y_2, y_3. Thus we may assume $y_3'' \in V(u_D X x_1 - \{u_D, x_1\})$.

Since X is induced and $H - X$ is connected, $H - D$ has a path R from y_3'' to B and internally disjoint from $B \cup X$.

We claim that R must end at y_3' and we may choose R to be a path of length at least 2. First, we may assume that $C' - y_1$ has disjoint paths L_1, L_2 from s, r to t, y_2, respectively; for otherwise, $(C' - y_1, s, r, t, y_2)$ is planar by Corollary 2.3, and hence G contains TK_5 by Corollary 2.9 (since $|V(C)| \geq 2$ by Lemma 3.1). If $G[B \cup R + \{y_2, y_3\}]$ has disjoint paths M_1, M_2 from y_3'', y_3 to y_2, b_D, respectively, then $M_1 \cup y_3'' y_3 \cup y_3'' X x_1 \cup (y_3'' X u_D \cup R_1 \cup L_1 \cup R_2 \cup v_D X x_2) \cup (M_2 \cup R_3 \cup L_2) \cup K$ is a TK_5 in G with branch vertices $x_1, x_2, y_2, y_3, y_3''$. If $G[B \cup R + \{y_2, y_3\}]$ has disjoint paths N_1, N_2 from y_3'', y_3 to b_D, y_2, respectively, then $(N_1 \cup R_3 \cup L_2) \cup y_3'' y_3 \cup y_3'' X x_1 \cup (y_3'' X u_D \cup R_1 \cup L_1 \cup R_2 \cup v_D X x_2) \cup N_2 \cup K$ is a TK_5 in G with branch vertices $x_1, x_2, y_2, y_3, y_3''$. So we may assume that M_1, M_2 do not exist, and N_1, N_2 do not exist. Therefore, R must end at y_3'. Moreover, we may choose R to be a path of length at least 2; as otherwise there are two edges from y_3'' to B, and M_1, M_2 or N_1, N_2 would exist.

Note that $R - y_3'$ is contained in a 2-connected block F of $H - B$, and let b_F, c_F, u_F, v_F be defined as before; so $y_3' \in \{b_F, c_F\}$. Then we may assume that $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order. This completes the proof of (8).

By (7) and (8), let $w \in N(y_3) \cap V(F - \{u_D, x_1\})$. We may assume that

(9) $w \notin \{u_F, v_F\}$.

Suppose $w \in \{u_F, v_F\}$ for any choice of w. Then $y_3 \notin N(F - \{u_F, v_F\})$. Hence we may assume that $y_1, y_2 \in N(F - \{u_F, v_F\})$, which follows from 5-connectedness of G when $b_F = c_F$, or from the planarity of (F', b_F, u_F, c_F, v_F) when $b_F \neq c_F$ (as otherwise G contains TK_5 by Corollary 2.9 and Lemma 3.1).

Let $S' := \{b_F, c_F, u_F, v_F, y_1, y_2\}$. Since $G[F + y_1]$ is connected, it contains three independent paths from some vertex $u \in V(F) - \{u_F, v_F\}$ to u_F, v_F, y_1, respectively. Since $G[F' + \{y_1, y_2\}]$ is $(5, S')$-connected, it follows from Lemma 2.4 that $G[F' + \{y_1, y_2\}]$ contains five independent paths W_1, W_2, W_3, W_4, W_5 from u to S' such that $V(W_i \cap W_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(W_i) \cap S'| = 1$ for $1 \leq i \leq 5$, $u_F \in V(W_1)$, $v_F \in V(W_2)$, and $y_1 \in V(W_3)$. Without loss of generality, we may assume that W_4 ends in $\{b_F, c_F\}$. Thus W_4 can be extended through $G[B + y_2]$ to a path W_4' ending at y_2.

If $C' - r$ contains disjoint paths L_1, L_2 from y_1, s to y_2, t, respectively, then $W_3 \cup W_4' \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X u_D \cup R_1 \cup L_2 \cup R_2 \cup v_D X x_2) \cup L_1 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume that L_1, L_2 do not exist in $C' - r$. By Corollary 2.3, $(C' - r, y_1, s, y_2, t)$ is planar; so G contains TK_5 by Corollary 2.9 (as $|V(C)| \geq 2$ by Lemma 3.1). This proves (9).

By (9), $w \in V(F) - \{u_F, v_F\}$. Let $S' := \{b_F, c_F, u_F, v_F\} \cup (N(F - \{u_F, v_F\}) \cap \{y_1, y_2\})$. It is clear that $G[F' + S']$ is $(4, S')$-connected. Also note that F has independent paths from w to u_F, v_F, respectively (as F is 2-connected). So by Lemma 2.4, $G[F' + S']$ contains four independent paths W_1, W_2, W_3, W_4 from w to S' such that $V(W_i \cap W_j) = \{u\}$ for $1 \leq i < j \leq 4$, $|V(W_i) \cap S'| = 1$ for $1 \leq i \leq 4$, $u_F \in V(W_1)$, and $v_F \in V(W_2)$. Without loss of generality, we may assume that $b_F = y_3'$ and $c_F \notin V(W_3)$.

If W_3 or W_4, say W_3, ends at y_2, then $wy_3 \cup W_3 \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_2, y_3. (Recall that Q is given before (5).)

Now assume that W_3 or W_4, say W_3, ends at y_1. If $C' - y_2$ has disjoint paths L_1, L_2 from r, s to y_1, t, respectively, then let Q' denote a path in $G[B + y_3]$ between b_D and y_3; so $wy_3 \cup W_3 \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X u_D \cup R_1 \cup L_2 \cup R_2 \cup v_D X x_2) \cup (Q' \cup R_3 \cup L_1) \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_3. So we may assume that L_1, L_2 do not exist. Then by Corollary 2.3, $(C' - y_2, r, s, y_1, t)$ is planar; so G contains TK_5 by Corollary 2.9 (as $|V(C)| \geq 2$ by Lemma 3.1).

We may thus assume that W_3 ends at $b_F = y_3'$ and W_4 ends at c_F (so $b_F \neq c_F$). Suppose $N(y_2 \cap V(B) \neq \{b_D\}$. In $G[B + y_2]$ we find independent paths Q_1, Q_2 from b_F to b_D, y_2, respectively. Then $(y_3 y_3' \cup W_3 \cup y_3 w) \cup (x_1 y_2 \cup Q_2) \cup x_1 y_3 \cup (x_1 X u_F \cup W_1) \cup (x_1 y_1 \cup P_1) \cup (P_5 \cup R_3 \cup Q_1) \cup (P_3 \cup R_1 \cup u_D X v_F \cup W_2) \cup (P_4 \cup R_2 \cup v_D X x_2 \cup x_2 y_3)$ is a TK_5 in G with branch vertices w, x_1, y_3, y_3', z.

So assume $N(y_2) \cap V(B) = \{b_D\}$. If $x_1 \neq u_F$ then G has a path Z from x_1 to $b \in V(B)$ and internally disjoint from $B \cup D \cup X$; and if $x_1 = u_F$ then Z can be chosen so that $b = c_F \neq b_F$. If $b \neq b_F$ then let Z_1, Z_2 be independent paths from b_D to b, b_F, respectively; then $(P_2 \cup P_5 \cup R_3 \cup y_2 b_D) \cup (P_3 \cup R_1 \cup u_D X x_1) \cup (P_4 \cup R_2 \cup v_D X x_2) \cup (Z_1 \cup Z) \cup (Z_2 \cup b_F y_3 x_2) \cup (K - y_3)$ is a TK_5 in G with branch vertices z, x_1, x_2, b_D, y_2. So assume $b = b_F$. Now $(wy_3 b_F \cup W_3) \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X x_2) \cup Z \cup (Q \cup b_D y_2 x_2) \cup (K - y_2)$ is a TK_5 in G with branch vertices z, x_1, x_2, b_F, y_3. ∎

Let D^* be obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 to a single vertex y, and let $A^* := \{y, b_D, c_D, u_D, v_D\}$. Recall that D' does not contain edges from $\{b_D, c_D\}$ to $\{u_D, v_D\}$, and

$$(D^* - y, b_D, u_D, c_D, v_D) \text{ is planar.}$$

Since D is of type I, $|V(D) - \{b_D, c_D\}| \geq 2$. So we may assume

$$|N(D - \{u_D, v_D\}) \cap \{y_1, y_2, y_3\}| \geq 2;$$

as otherwise, G contains TK_5 by Corollary 2.9. By Lemma 3.1, we may assume $|V(D)| \geq 4$; so $|V(D^*)| \geq 7$. By Lemma 4.2, we may assume that

$$D^* \text{ is } (5, A^*)\text{-connected.}$$

Let C denote the facial walk of $D^* - y$ containing $A^* - \{y\}$ and assume that it is the outer walk of $D^* - y$. Then C is a cycle, or b_D (or c_D) has degree 1 in C and $C - b_D$ (or $C - c_D$) is a cycle, or b_D, c_D both have degree 1 in C and $C - \{b_D, c_D\}$ is a cycle.

We now show that there exist paths in D^* as shown in Corollaries 2.11 and 2.12.

Lemma 4.3 *G contains TK_5, or there exist $w \in V(D^*) - A^*$ and a cycle C_w in $D^* - y$ such that C_w consists of all vertices of $D^* - y$ cofacial with w, and one of the following holds:*

(1) *w is a neighbor of y and $D^* - y$ has three independent paths P_1, P_2, P_3 from w to $\{b_D, c_D, u_D, v_D\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A^*| = 1$ for $i = 1, 2, 3$;*

(2) *y has no neighbor in D^*-C, $C\cap C_w=\emptyset$, and D^* has four independent paths P_1,P_2,P_3,P_4 from w to A^* such that $V(P_i\cap P_j)=\{w\}$ for $1\le i<j\le 4$, $|V(P_i\cap C_w)|=|V(P_i)\cap A^*|=1$ for $1\le i\le 4$, and either (a) $y\notin\bigcup_{i=1}^4 V(P_i)$, or (b) $y\in\bigcup_{i=1}^4 V(P_i)$ and we can write $A^*-\{y\}=\{a_1,a_2,a_3,a_4\}$ such that $y\in V(P_1)$, $a_i\in V(P_i)$ for $i=2,3,4$, $a_1,a_2,a_3,V(P_1\cap C),a_4$ occur on C in cyclic order.*

Proof. If D^* has a 5-separation (F_1,F_2) such that $\{b_D,c_D,u_D,v_D,y\}\subseteq V(F_1)$ and $|V(F_2)|\ge 7$, and we choose (F_1,F_2) so that F_2 is minimal and let $A:=V(F_1)\cap V(F_2)$; otherwise let $F_2=D^*$ and $A:=\{b_D,c_D,u_D,v_D,y\}$. By the minimality of F_2, A is independent in F_2 and F_2-y is 2-connected. Let C' denote the the outer cycle of F_2-y, which contains $A-y$.

We may assume $y\in A$; for, otherwise, (F_2,A) is planar, and hence G contains TK_5 by Lemma 2.6. We may also assume D^*-y contains no K_4^- as otherwise G contains K_4^-, and hence G contains TK_5 by Theorem 1.1.

By Menger's theorem, there are four disjoint paths in F_1-y from $A-\{y\}$ to $A^*-\{y\}$, which allows us to extend the paths we will find in F_2 to the desired paths in D^*.

If y has a neighbor inside C', say w, then (1) follows from Corollary 2.11 (after appropriate extension of the paths to A^*). Hence we may assume that C' contains all neighbors of y in F_2. If F_2 is not the exceptional graph in Corollary 2.12, then (2) follows from Corollary 2.12 (after appropriate extensions of the paths to A^*).

So we may assume that F_2 is the exceptional graph. Let $A=\{b',c',u',v'\}$ and $tuvwt$ be the cycle in F_2-A such that $C'=b'tv'uc'vu'wb'$, and let Q_1,Q_2,Q_3,Q_4 be disjoint paths in F_1-y from b',c',u',v' to b_D,c_D,u_D,v_D, respectively.

Since G is 5-connected and by Lemma 3.1, we may assume that each of $\{t,u,v,w\}$ has exactly one neighbor in $\{y_1,y_2,y_3\}$. Since G contains no K_4^-, we may assume by symmetry that $y_3\in N(u)\cap N(w)$ and that either $y_2\in N(t)\cap N(v)$ or $y_1\in N(v)$ and $y_2\in N(t)$.

Suppose $y_2\in N(t)\cap N(v)$. Then by Lemma 3.1, we may assume $y_1\notin N(\{t,u,v,w\})$. Note that $G':=G-\{t,u,v,w,y_2,y_3\}$ contains two paths R_1,R_2 from b' to $\{c',u',v'\}$ such that $R_1\cap R_2=\{b'\}$; for otherwise, G' has a cut T, $|T|\le 1$, separating b' from $\{c',u',v'\}$, and so $\{b',y_2,y_3\}\cup T$ would be a cut in G, contradicting 5-connectedness of G. Clearly, R_1,R_2 can be extended, using $u'v$ or $c'v$ and $v'u$ or $c'u$, to give independent paths R_1',R_2' in $G-\{t,w,y_2,y_3\}$ from b' to u,v, respectively. Now $b't\cup b'w\cup R_1'\cup R_2'\cup tuvwt\cup ty_2v\cup uy_3w$ is a TK_5 in G with branch vertices b',t,u,v,w.

Thus we may assume that $y_1\in N(v)$ and $y_2\in N(t)$. Note that the triangle $b'twb'$ is contained in a block of $H-(x_1Xu_D\cup Q_3\cup u'vuv'\cup Q_4\cup v_DXx_2)$ and has two neighbors in $\{y_1,y_2,y_3\}$. So by Lemma 3.2 and by the choice of X, we may assume $|N(B)\cap\{y_1,y_2,y_3\}|\ge 2$. If $y_1,y_2\in N(B)$ then let Q be a path in $G[B+\{y_1,y_2\}]$ between y_1 and y_2; now $(twu'\cup Q_3\cup u_DXx_1)\cup(tv'\cup Q_4\cup v_DXx_2)\cup(tuvy_1)\cup ty_2\cup Q\cup K$ is a TK_5 in G with branch vertices t,x_1,x_2,y_1,y_2. So by symmetry we may assume that $y_2,y_3\in N(B)$. Let R denote a path in $G[B+\{y_2,y_3\}]$ between y_2 and y_3. Then $(tuvu'\cup Q_3\cup u_DXx_1)\cup(tv'\cup Q_4\cup v_DXx_2)\cup ty_2\cup twy_3\cup R\cup K$ is a TK_5 in G with branch vertices t,x_1,x_2,y_2,y_3. ∎

Lemma 4.4 *Suppose D^* contains w,C_w,P_1,P_2,P_3 which satisfy* (1) *of Lemma 4.3. Then G contains TK_5.*

Proof. Without loss of generality, we may assume that $y_1w\in E(G)$. Let $L=C_w\cup P_1\cup P_2\cup P_3\cup y_1w$. We may assume that

(1) $\{b_D, c_D\} \subseteq V(L)$, and $v_D \in V(L)$ (by symmetry).

If $\{b_D, c_D\} \subseteq V(L)$, then (1) holds by letting $v_D \in V(L)$ using symmetry between u_D and v_D. So assume $\{u_D, v_D\} \subseteq V(L)$. By symmetry, we may assume $b_D \in L$.

We may assume that $x_1 = u_D$ and $x_2 = v_D$. Otherwise, we may assume by symmetry that $x_1 \neq u_D$. Then H has a path Q from x_1 to b_D and internally disjoint from $X \cup D'$. Now $Q \cup x_1y_1 \cup x_1Xu_D \cup (x_1y_2x_2 \cup x_2Xv_D) \cup L$ is a TK_5 in G.

We may further assume that $x_1, x_2 \in V(C_w)$. For, suppose not. Then by symmetry assume $x_1 \notin V(C_w)$. Since G is 5-connected and $u_D = x_1$ and $v_D = x_2$, there exists $i \in \{1, 2, 3\}$ such that $y_i \in N(B - \{b_D, c_D\})$. Let R be a path in $G[B + y_i]$ from y_i to b_D. If $i = 1$ then $R \cup y_1x_1 \cup y_1x_2 \cup L$ is a TK_5 in G. So we may assume $i \in \{2, 3\}$. Then $x_1y_1 \cup (x_1y_i \cup R) \cup x_1y_{5-i}x_2 \cup L$ is a TK_5 in G.

If $|V(x_2C_wx_1)| = 2$ then $x_1x_2 \in E(G)$; so $G[x_1, x_2, y_1, y_2] \cong K_4^-$, and G contains TK_5 by Theorem 1.1. So we may assume that $|V(x_2C_wx_1)| \geq 3$.

Suppose w has no neighbor in $x_2C_wx_1 - \{x_1, x_2\}$. Since D^* is $(5, A^*)$-connected, $\{x_1, x_2, c_D\}$ cannot be a cut in D separating $\{b_D, c_D, x_1, x_2\}$ from some vertex. Therefore, $x_2C_wx_1 = x_2c_Dx_1$. As D is of type I, $c_Dw \in E(G)$, a contradiction.

Therefore, we may assume that w has a neighbor $w' \in V(x_2C_wx_1) - \{x_1, x_2\}$. If D contains a path Q from w' to c_D and internally disjoint from C_w, then replacing the path in L from w to u_D with $Q + \{w, ww'\}$ we get (1). So we may assume that such Q does not exist. Then since $(D^* - y, b_D, u_D, c_D, v_D)$ is planar, there exist $u \in V(w'C_wx_1 - w')$ and $v \in V(x_2C_ww' - w')$ such that $\{u, v, w\}$ is a cut in D separating $\{b_D, c_D, x_1, x_2\}$ from w', contradicting the fact that D^* is $(5, A^*)$-connected. This proves (1).

(2) We may assume $x_1 \notin V(C_w)$.

For, suppose $x_1 \in V(C_w)$. Then $x_1y_1 \cup (x_1y_2x_2 \cup x_2Xv_D) \cup L$ and a path in B between b_D and c_D form a TK_5 in G with branch vertices w, x_1 and $P_i \cap C_w$, $i = 1, 2, 3$. So we have (2).

(3) We may assume that $D - u_D$ and $D - v_D$ are 2-connected, and $D' - \{u_D, v_D\}$ is a chain of blocks from b_D to c_D.

First, suppose $D - u_D$ is not 2-connected. Then let J be an endblock of $D - u_D$ and v be the cut vertex of $D - u_D$ contained in J such that $v_D \notin V(J - v)$. Since D is 2-connected, $u_D \in N(J - v)$ and $u_D \in N(D - u_D - J)$. In particular, $D - (J - v)$ contains a path from u_D to v_D. Thus, since (D', b_D, u_D, c_D, v_D) is planar, $b_D \notin N(J - v)$ or $c_D \notin N(J - v)$, say the former. Then $\{c_D, u_D, v\}$ is a cut in D' separating J from $\{b_D, c_D, u_D, v_D\}$, contradicting the assumption that D^* is $(5, A^*)$-connected.

Thus we may assume that $D - u_D$ is 2-connected. Similarly, we may also assume that $D - v_D$ is 2-connected.

By the definition of 3-planar chain, $D - \{u_D, v_D\}$ is connected. So $D' - \{u_D, v_D\}$ is connected. Now suppose $D' - \{u_D, v_D\}$ is not a chain of blocks from b_D to c_D. Then let J be an endblock of $D' - \{u_D, v_D\}$ and v be the cut vertex of $D' - \{u_D, v_D\}$ such that $D' - \{u_D, v_D\} - (J - v)$ has a path between b_D and c_D. Then $\{u_D, v_D, v\}$ is a cut in D' separating J from $\{b_D, c_D, u_D, v_D\}$, contradicting the assumption that D^* is $(5, A^*)$-connected.

(4) We may assume $u_D = x_1$, and H contains no path from x_1 to B internally disjoint from $B \cup D' \cup X$.

Suppose (4) fails. Note that if $u_D \neq x_1$ then H contains a path from x_1 to B internally disjoint from $B \cup D' \cup X$. So let R be an arbitrary path in H from x_1 to $x \in V(B)$ and internally disjoint from $B \cup D' \cup v_D X x_2$.

Suppose x may be chosen so that there exists some $y_i \in N(B - x)$. Then $G[B \cup R + y_i]$ contains disjoint paths Q_1, Q_2 from $\{b_D, c_D\}$ to x_1, y_i, respectively. Recall $x_1 \notin V(C_w)$ from (2). If $i = 1$ then $(y_1 x_1 \cup Q_1) \cup Q_2 \cup (y_1 x_2 \cup x_2 X v_D) \cup L$ is a TK_5 in G. So assume $i \neq 1$. Then $Q_1 \cup (x_1 y_i \cup Q_2) \cup x_1 y_1 \cup (x_1 y_{5-i} x_2 \cup x_2 X v_D) \cup L$ is a TK_5 in G.

Therefore, we may assume that x is unique and $y_i \notin N(B - x)$ for all $i = 1, 2, 3$. So by Lemma 3.1, we may assume $|N(B) \cap \{y_1, y_2, y_3\}| \leq 1$.

Suppose H has a path from $x_2 X v_D$ to B internally disjoint from $B \cup D' \cup X$. Then we may assume $x_1 = u_D$; otherwise H has a path from x_1 to x_2 and disjoint from $D - v_D$, and hence by Lemma 3.2 and the choice of X, G contains TK_5 (or else $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(D - v_D) \cap \{y_1, y_2, y_3\}| \geq 2$ by (3), a contradiction). Similarly, we may assume $x_2 = v_D$. Since G is 5-connected, $V(B) = \{b_D, c_D, x\}$ and x is adjacent to x_1, x_2 and some y_i. So $G[\{x, x_1, x_2, y_i\}] \cong K_4^-$ and, by Theorem 1.1, G contains TK_5.

Thus we may assume that H has no path from $x_2 X v_D$ to B internally disjoint from $B \cup D' \cup X$; so $x_2 = v_D$. Since $\{b_D, c_D, u_D, x\}$ cannot be a cut in G, we see that $|B| = 3$ and $x \notin \{b_D, c_D\}$. Since x has at least three neighbors outside B, $G - D'$ contains independent paths Q_1, Q_2 from x to x_1, y_i, respectively, for some $i \in \{1, 2, 3\}$. If $i = 1$ then $(Q_1 \cup x_1 y_2 x_2) \cup Q_2 \cup (B - b_D c_D) \cup L$ is a TK_5 in G; and if $i \neq 1$ then $(Q_1 \cup x_1 y_2) \cup (Q_2 \cup y_i x_2 \cup x_2 X v_D) \cup (B - b_D c_D) \cup L$ is a TK_5 in G. Hence we have (4).

(5) We may assume that $y_1 \notin N(B - \{b_D, c_D\})$ and $|N(y_1) \cap B| \leq 1$.

First, suppose $|N(y_1) \cap B| \geq 2$. Then $G[B + y_1]$ has two independent paths Q_1, Q_2 from y_1 to b_D, c_D, respectively. So $Q_1 \cup Q_2 \cup (y_1 x_2 \cup x_2 X v_D) \cup L$ is a TK_5 in G.

Now let $\{y_1'\} = N(y_1) \cap V(B - \{b_D, c_D\})$. Since G is 5-connected, $G - (D - v_D)$ has a path from $B - \{b_D, c_D\}$ to $x_2 X v_D + \{y_2, y_3\}$. If $G - (D - v_D) - y_1$ has three independent paths Q_1, Q_2, Q_3 from y_1' to b_D, c_D, v_D, respectively, then $Q_1 \cup Q_2 \cup Q_3 \cup y_1' y_1 \cup L$ is a TK_5 in G. So we may assume that such Q_1, Q_2, Q_3 do not exist. Then there is a 2-cut S in $G - (D - v_D) - y_1$ separating y_1' from $\{b_D, c_D, v_D\}$. Since B is 2-connected, $S \subseteq V(B)$. But then by (4), $S \cup \{x_1, y_1\}$ is a 4-cut in G, a contradiction which completes the proof of (5).

Let $S := \{b_D, c_D, y_2, y_3\} \cup V(x_2 X v_D)$. Then by (4) and (5), $G' := G - y_1 - (D - v_D)$ is $(5, S)$-connected and, since $H - X$ is connected, $G' - \{y_2, y_3\}$ contains a path from $B - \{b_D, c_D\}$ to some $v \in V(x_2 X v_D)$ and internally disjoint from X. We choose v so that $v X v_D$ is minimal. Note that $G' - \{y_2, y_3\} - (x_2 X v_D - v)$ has independent paths from some $u \in V(B) - \{b_D, c_D\}$ to b_D, c_D, v, respectively. So by Lemma 2.4, G' contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to b_D, c_D, v, z_1, z_2, respectively, where $z_1, z_2 \in S - \{b_D, c_D, v\}$ such that $|V(Q_i) \cap S| = 1$ for $1 \leq i \leq 5$. If $v \neq x_2$ then Q_4 can be extended through $G[(x_2 X v - v) + \{y_1, y_2, y_3\}]$ to a path Q_4' ending at y_1; so $Q_1 \cup Q_2 \cup (Q_3 \cup v X v_D) \cup Q_4' \cup L$ is a TK_5 in G. So assume $v = x_2$. Then by the minimality of $v X v_D$, we see that $z_1 \in \{y_2, y_3\}$, say $z_1 = y_2$. Now by (2), $Q_1 \cup Q_2 \cup (Q_3 \cup v X v_D) \cup (Q_4 \cup y_2 x_1 y_1) \cup L$ is a TK_5 in G. ∎

Lemma 4.5 *Suppose D^* contains $w, C_w, P_1, P_2, P_3, P_4$ which satisfy* (2) *of Lemma 4.3. Then G contains TK_5.*

Proof. Let $L = C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4$. If $y \notin V(L)$ then L, $u_D X x_1 \cup x_1 y_1 x_2 \cup x_2 X v_D$ and a path in B between b_D and c_D form a TK_5 in G. So we may assume $y \in V(P_1)$ and, without loss of generality, we may view P_1 as a path in G with $y_1 \in V(P_1)$.

We may assume $b_D, c_D \in V(L)$. For, suppose $u_D, v_D \in V(L)$. By symmetry, we may assume that $u_D \in V(P_2)$, $b_D \in V(P_3)$, $v_D \in V(P_4)$, and $c_D, u_D, b_D, V(P_1 \cap C), v_D$ occur on C in clockwise order. If $P_2 \cap v_D C c_D \neq \emptyset$ then by planarity we may modify P_2 to end at c_D; so we could assume $b_D, c_D \in V(L)$. Hence we may assume that $P_2 \cap v_D C c_D = \emptyset$. Then, since $C \cap C_w = \emptyset$, there exists a path P_4' in $P_4 \cup v_D C c_D$ from w to c_D such that $|V(P_4') \cap V(C_w)| = 1$. Let $L' = C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4'$, and let Q be a path in B between b_D and c_D. Now $Q \cup (y_1 x_1 \cup x_1 X u_D) \cup L'$ is a TK_5 in G.

By symmetry, we may assume that $b_D \in V(P_2)$, $v_D \in V(P_3)$, $c_D \in V(P_4)$, and $u_D, c_D, v_D, V(P_1 \cap C), b_D$ occur on C in counterclockwise order.

Recall that $C_w \cap C = \emptyset$. Then $P_4 \cup c_D C u_D$ contains a path P_4' from w to u_D such that $|V(P_4' \cap C_w)| = 1$; and in this case we let $L' = C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4'$. Similarly, $P_2 \cup u_D C b_D$ contains a path P_2' from w to u_D such that $|V(P_2' \cap C_w)| = 1$; and in this case we let $L'' = C_w \cup P_1 \cup P_2' \cup P_3 \cup P_4$.

We may assume that H contains no path from $x_2 X v_D + \{y_2, y_3\}$ to $B - \{b_D, c_D\}$ and internally disjoint from $B \cup D \cup X$. For, otherwise, H contains a path Q from v_D to b_D and disjoint from $(D - v_D) \cup x_1 X u_D + c_D$. Then $Q \cup (y_1 x_1 \cup x_1 X u_D) \cup L'$ is a TK_5 in G.

Thus $x_2 = v_D$. Moreover, $x_1 \neq u_D$; otherwise $\{b_D, c_D, x_1, y_1\}$ would be a 4-cut in G. So H has a path X_1 from x_1 to b_D and internally disjoint from $D \cup X$.

If H has a path Q from y_1 to some $y_1' \in V(u_D X x_1 - x_1)$ and internally disjoint from $B \cup D' \cup X$, then $(Q \cup y_1' X u_D) \cup (X_1 \cup x_1 y_2 x_2) \cup L'$ is a TK_5 in G; so we may assume that $N(y_1) \cap (V(H - x_1) - V(B \cup (D - u_D))) = \emptyset$. If H has a path Q from y_1 to c_D and disjoint from $D \cup X \cup \{b_D\}$ then $Q \cup (x_2 y_2 x_1 \cup x_1 X u_D) \cup L''$ is a TK_5 in G; so we may assume $N(y_1) \cap V(B - c_D) = \emptyset$. Hence $N(y_1) \subseteq V(D' - u_D) \cup \{x_1, x_2\}$.

Thus, let $y' \in N(\{y_2, y_3\}) \cap V(u_D X x_1 - x_1)$ with $u_D X y'$ minimal; so $y' \in N(y_i)$ for some $i \in \{2, 3\}$. Since $\{b_D, c_D, u_D, y'\}$ cannot be a 4-cut in G, H has a path from $y' X x_1 - y'$ to $B - b_D$ and internally disjoint from $B \cup D \cup X$. Thus H has a path R from x_1 to c_D and internally disjoint from $y' X x_2 \cup D$. Now $(x_2 y_i y' \cup y' X u_D) \cup (y_1 x_1 \cup R) \cup L''$ is a TK_5 in G. ∎

We can now summarize the results in this section as the following

Lemma 4.6 *If some block in $H - B$ is of type I then G contains TK_5.*

5 Blocks of type II

In this section we show, with the help of Lemma 4.6, that if $H - B$ has a block of type II then G contains TK_5. Let D be a block of $H - B$ of type II, and recall the notation D', b_D, u_D, v_D (in particular, x_1, u_D, v_D, x_2 occur on X in this order). Let $D'' := D - \{u_D, v_D\}$ which is connected by the definition of a 3-planar chain. Since G is 5-connected and D is of type II, $|N(D'') \cap \{y_1, y_2, y_3\}| \geq 2$. An important step is to show that $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

Lemma 5.1 *If $H-B$ has a block of type II then G contains TK_5 or $|N(B)\cap\{y_1,y_2,y_3\}|\geq 2$.*

Proof. First, we may assume $K_4^- \not\subseteq G$, as otherwise G contains TK_5 by Theorem 1.1.

(1) We may assume that D'' or $G[D''+b_D]$ is 2-connected.

Since $|N(D'')\cap\{y_1,y_2,y_3\}|\geq 2$, we have $|V(D'')|\geq 2$ by Lemma 3.1. In fact, $|V(D'')|\geq 3$ as D is 2-connected and $K_4^- \not\subseteq G$. Let $C_1,\dots,C_k$ denote the endblocks of D''. We may assume $k\geq 2$, as otherwise D'' is 2-connected and hence (1) holds. Let $v_i\in V(C_i)$ such that v_i is a cut vertex of D''.

If $\{u_D,v_D\}\not\subseteq N(C_i-v_i)$ for some $1\leq i\leq k$, then $\{u_D,v_i\}$ or $\{v_D,v_i\}$ is a cut in H separating B from some vertex; so G contains TK_5 by Lemma 3.4. Thus we may assume $u_D,v_D\in N(C_i-v_i)$ for $1\leq i\leq k$.

Suppose $|N(C_i)\cap\{y_1,y_2,y_3\}|\geq 2$ for some $1\leq i\leq k$. Then by Lemma 3.1, C_i is 2-connected. Let X' be obtained from X by replacing u_DXv_D with a path in $G[C_j+\{u_D,v_D\}]-v_j$ (for some $j\neq i$) between u_D and v_D. So C_i is contained in a 2-connected block of $H-X'$. Hence by Lemma 3.2 and by the choices of X, we may assume $|N(B)\cap\{y_1,y_2,y_3\}|\geq 2$.

Thus we may assume that for $1\leq i\leq k$, $|N(C_i)\cap\{y_1,y_2,y_3\}|\leq 1$. Then, since G is 5-connected, $b_D\in N(C_i-v_i)$ for $1\leq i\leq k$. So $G[D''+b_D]$ is 2-connected, and we have (1).

(2) We may assume that $D-u_D$ and $D-v_D$ are 2-connected.

Suppose $D-u_D$ is not 2-connected. Since D is 2-connected, $D-u_D$ is connected. Let C be an endblock of $D-u_D$ and let v be the cut vertex of $D-u_D$ such that $v_D\notin V(C-v)$. Then $\{u_D,v\}$ is a cut in H separating B from some vertex; so G contains TK_5 by Lemma 3.4. Hence we may assume $D-u_D$ is 2-connected. Similarly, we may assume that $D-v_D$ is 2-connected.

(3) We may assume $u_D\neq x_1$, $v_D=x_2$, and H contains no path from x_2 to B internally disjoint from $B\cup D'\cup X$.

If $u_D=x_1$ and $v_D=x_2$ then, since G is 5-connected, $|N(B-b_D)\cap\{y_1,y_2,y_3\}|\geq 2$. So we may assume by symmetry that $u_D\neq x_1$. Then, since H is 2-connected, H has a path from x_1 to B internally disjoint from $B\cup D'\cup X$.

Suppose H also has a path from x_2 to B internally disjoint from $B\cup D'\cup X$. Then H contains a path X' between x_1 and x_2 and disjoint from $D-v_D$. So by (2) and Lemma 3.2 and by the choice of X, we may assume $|N(B)\cap\{y_1,y_2,y_3\}|\geq|N(D-v_D)\cap\{y_1,y_2,y_3\}|\geq 2$.

So we may assume $x_2=v_D$ and H contains no path from x_2 to B internally disjoint from $B\cup D'\cup X$. This proves (3).

Since D'' is connected, we have

(4) for any $y_i,y_j\in N(D'')$, $G[D''+\{x_2,y_i,y_j\}]$ contains three independent paths from some vertex $u\in V(D'')$ to x_2,y_i,y_j, respectively.

By (3), there are at most two 2-connected blocks in $H-B$. So we have two cases.

Case 1. D is the unique 2-connected block in $H-B$.

By Lemma 3.1 and since G is 5-connected, $|N(x) \cap V(B)| \geq 2$ for each $x \in V(x_1 X u_D) - \{x_1, u_D\}$.

Subcase 1.1. $N(y_i) \subseteq V(D') \cup \{x_1, x_2\}$ for some $i \in \{1,2,3\}$, say $i = 1$.

Then $\{b_D, u_D, x_1, y_2, y_3\}$ is a cut in G. Let $G_1 := G - (D'' + \{x_2, y_1\})$. By Lemma 3.1, $|V(G_1)| \geq 7$.

Suppose $y_2, y_3 \in N(D'')$. Then by (4) (with $\{i,j\} = \{2,3\}$) and by Lemma 2.4, there exist four independent paths P_1, P_2, P_3, P_4 in $G[D' + \{y_2, y_3\}]$ from some vertex $u \in D''$ to $x_2, y_2, y_3, s \in \{b_D, u_D\}$, respectively, such that $|V(P_i) \cap \{u_D, v_D, x_2, y_2, y_3\}| \leq 1$ for $1 \leq i \leq 4$. Let $t \in \{b_D, u_D\} - \{s\}$. If $G_1 - t$ has disjoint paths Q_1, Q_2 from x_1, y_2 to s, y_3, respectively, then $P_1 \cup (P_4 \cup Q_1) \cup P_2 \cup P_3 \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume that such paths do not exist. Then by Corollary 2.3, $(G_1 - t, x_1, y_2, s, y_3)$ is planar; and hence G contains TK_5 by Corollary 2.9.

So we may assume that $y_3 \notin N(D'')$. Then $\{b_D, u_D, x_2, y_1, y_2\}$ is a cut in G separating D'' from $B \cup u_D X x_1$.

We may assume that $G[D' + y_2]$ contains disjoint paths Q_1, Q_2 from u_D, b_D to x_2, y_2, respectively; for, otherwise, it follows from Corollary 2.3 that $(G[D' + y_2], u_D, b_D, x_2, y_2)$ is planar; so G contains TK_5 by Corollary 2.9 (as $|V(D'')| \geq 3$). Similarly, we may assume that $G[D' + y_2]$ contains disjoint paths Q_1', Q_2' from u_D, b_D to y_2, x_2, respectively.

Suppose $|N(y_3) \cap V(B)| \geq 2$. We may assume $y_2 \notin N(B)$, or else the assertion of the lemma holds. Hence y_2 has a neighbor $u \in V(u_D X x_1) - \{u_D, x_1\}$ (otherwise $\{x_1, b_D, u_D, y_3\}$ would be a 4-cut in G). Now $G[B + \{u, y_3\}]$ contains independent paths R_1, R_2 from y_3 to u, b_D, respectively, and $uy_2 \cup R_1 \cup uXx_1 \cup (uXu_D \cup Q_1) \cup (R_2 \cup Q_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Thus we may assume that $|N(y_1) \cap V(B)| \leq 1$. Hence, there exist distinct $v, v' \in N(y_3) \cap V(u_D X x_1 - x_1)$, and assume that x_1, v, v', u_D occur on X in order. We may assume that $y_2 \notin N(B - b_D)$; for otherwise $G[B + \{y_2, v\}]$ has independent paths R_1, R_2 from v to y_2, b_D, respectively, and $vy_3 \cup R_1 \cup vXx_1 \cup (R_2 \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_2, y_3. So y_2 has a neighbor $u \in V(u_D X x_1) - \{u_D, x_1\}$; as otherwise $\{b_D, u_D, x_1, y_3\}$ would be a 4-cut in G.

Suppose $u \in V(x_1 X v - v)$. Let R be a path in $G[B + u]$ from u to b_D. Then $uy_2 \cup (uXv \cup vy_3) \cup uXx_1 \cup (R \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Now assume $u \in V(vXv') - \{v, v'\}$. Then in $G[B + v]$ we find a path R from v to b_D. So $vy_3 \cup (vXu \cup uy_2) \cup vXx_1 \cup (R \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_2, y_3.

Therefore, we may assume by Lemma 3.1 that $u \in V(v' X u_D) - \{u_D, v'\}$. If $G[B + \{u, v, x_1\}]$ has disjoint paths R_1, R_2 from x_1, v to u, b_D, respectively, then $uy_2 \cup (uXv' \cup v'y_3) \cup R_1 \cup (uXu_D \cup Q_1) \cup (y_3 v \cup R_2 \cup Q_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume that R_1, R_2 do not exist. Then by Lemma 2.2, $(G[B + \{u, v, x_1\}], x_1, v, u, b_D)$ is 3-planar; so $G[B + \{u, v, x_1\}]$ contains disjoint paths L_1, L_2 from x_1, v to b_D, u, respectively. Hence $X' := Q_2' \cup L_1$ is a path in H between x_1 and x_2, and $uXv \cup L_2$ is a cycle in $H - X'$ and contains neighbors of both y_2 and y_3. So by Lemma 3.2 and by the choice of X we may assume $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

Subcase 1.2. $N(y_i) \not\subseteq V(D') \cup \{x_1, x_2\}$ for all $i = 1, 2, 3$.

We may assume $|N(B) \cap \{y_1, y_2, y_3\}| \leq 1$, as otherwise the assertion of the lemma holds. So by symmetry let $y_1, y_2 \notin N(B)$; hence $y_1, y_2 \in N(x_1 X u_D - \{x_1, u_D\})$. Further, if $y_3 \in N(x_1 X u_D - \{x_1, u_D\})$ then, by symmetry among y_1, y_2, y_3, we may assume that the neighbor of $\{y_1, y_2, y_3\}$ on $x_1 X u_D$ closest to u_D is a neighbor of y_3, denoted by v_3. Let $v_i \in N(y_i) \cap V(x_1 X u_D - \{x_1, u_D\})$, $i = 1, 2$. We may assume by symmetry between y_1 and y_2 that x_1, v_1, v_2, u_D occur on X in order. Note that $|N(v_i) \cap V(B)| \geq 2$ for $i = 1, 2$. Let X_1 denote a path in $G[B + x_1]$ from x_1 to b_D.

We may assume $y_3 \in N(D'')$. For, suppose $y_3 \notin N(D'')$. Then $\{b_D, u_D, x_2, y_1, y_2\}$ is a 5-cut in G separating D' from $B \cup u_D X x_1$. In $G[D' + \{y_1, y_2\}]$ we apply Menger's theorem to find five independent paths P_1, P_2, P_3, P_4, P_5 from some $u \in V(D'')$ to y_1, y_2, x_2, b_D, u_D, respectively. Now $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_1 v_1 \cup v_1 X v_2 \cup v_2 y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

We may also assume $y_1, y_2 \in N(D'')$. For, suppose $y_1 \notin N(D'')$. Then $y_2, y_3 \in N(D'')$ as G is 5-connected, and $\{b_D, u_D, x_2, y_2, y_3\}$ is a cut in G separating D' from $B \cup u_D X x_1$. By Menger's theorem, $G[D' + \{y_2, y_3\}]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in V(D'')$ to y_2, y_3, x_2, b_D, u_D. If v_3 is defined then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_2 v_2 \cup v_2 X v_3 \cup v_3 y_3) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume that v_3 is not defined. Thus $y_3 \in N(B)$ (as we are in Subcase 1.2). Moreover, $G[B + \{v_2, y_3\}]$ contains a path R from v_2 to y_3. If $G[D' + \{y_2, y_3\}] - b_D$ has disjoint paths Q_1, Q_2 from u_D, y_2 to x_2, y_3, respectively, then $v_2 y_2 \cup R \cup v_2 X x_1 \cup (v_2 X u_D \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_2, y_3. So assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D' + \{y_2, y_3\}] - b_D, u_D, y_2, x_2, y_3)$ is planar; so G contains TK_5 by Corollary 2.9. Hence, we may assume $y_1 \in N(D'')$. Similarly, we may assume $y_2 \in N(D'')$.

Let $S := \{b_D, u_D, x_2, y_1, y_2, y_3\}$. By (4) (with $\{i, j\} = \{1, 2\}$) and by Lemma 2.4, $G[D'' + S]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in V(D'')$ to S such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in V(P_1)$, $y_2 \in V(P_2)$, and $x_2 \in V(P_3)$. We may assume that P_4 ends in $\{b_D, u_D\}$. We may further assume that P_4 ends at u_D; or else, $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_1 v_1 \cup v_1 X v_2 \cup v_2 y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

We may further assume that v_3 is not defined. For, otherwise, $v_3 \in V(u_D X v') - \{u_D, v'\}$ by the definition of v_3. Let X_1' be a path in $G[B + \{v_3, x_1\}]$ from x_1 to v_3. Then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup u_D X v_3 \cup X_1') \cup (y_1 v_1 \cup v_1 X v_2 \cup v_2 y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So $y_3 \in N(B - b_D)$ since $N(y_3) \not\subseteq V(D') \cup \{x_1, x_2\}$. Let D^* be obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying u_D and b_D as w.

Suppose $D^* - y_3$ contains disjoint paths Q_1, Q_2 from y_1, w to y_2, x_2, respectively. We view Q_2 as a path in G; so $u_D \in V(Q_2)$ or $b_D \in V(Q_2)$. If $b_D \in V(Q_2)$ then let Q be a path in $G[B + v_1]$ from v_1 to b_D; now $v_1 y_1 \cup (v_1 X v_2 \cup v_2 y_2) \cup v_1 X x_1 \cup (Q \cup Q_2) \cup Q_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_2. So we may assume $u_D \in V(Q_2)$. Let R be a path in $G[B + \{v_2, x_1\}]$ from v_2 to x_1. Then $v_2 y_2 \cup (v_2 X v_1 \cup v_1 y_1) \cup R \cup (v_2 X u_D \cup Q_2) \cup Q_1 \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_1, y_2.

Therefore, we may assume that such Q_1, Q_2 do not exist in $D^* - y_3$. So by Lemma 2.2, $(D^* - y_3, y_1, w, y_2, x_2)$ is 3-planar. Since D is 2-connected, $D^* - \{y_1, y_2, y_3\}$ is 2-connected. Thus, $D^* - y_1$ contains disjoint paths R_1, R_2 from y_2, x_2 to y_3, w, respectively, or $D^* - y_2$ contains disjoint paths R_1, R_2 from y_1, x_2 to y_3, w, respectively. By symmetry between y_1 and

y_2, we may assume the latter. We view R_2 as a path in G; so $b_D \in V(R_2)$ or $u_D \in V(R_2)$. Note that $G[B + \{v_1, y_3\}]$ contains independent paths L_1, L_2 from v_1 to y_3, b_D, respectively. If $b_D \in V(R_2)$, then $v_1y_1 \cup L_1 \cup v_1Xx_1 \cup (L_2 \cup R_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_3. So we may assume $u_D \in V(R_2)$. Then $v_1y_1 \cup L_1 \cup v_1Xx_1 \cup (v_1Xu_D \cup R_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_3.

Case 2. $H - B$ has a 2-connected block D_1 such that $D_1 \neq D$.

Then by (3), $u_{D_1} = x_1$, and H contains no path from x_1 to B internally disjoint from $B \cup F' \cup X$. Hence D_1 and $D_2 := D$ are the only 2-connected blocks of $H - B$. For $i = 1, 2$, let $D_i'' = D_i - \{u_D, v_D\}$ and $b_i := b_{D_i}$, and let $v_1 := v_{D_1}$ and $u_2 := u_{D_2}$. We may assume $|N(B) \cap \{y_1, y_2, y_3\}| \leq 1$, or else we have the assertion of this lemma. So, since G is 5-connected, there is an edge between v_1Xu_2 and $B - \{b_1, b_2\}$. Moreover, if $b_1 = b_2$ then $x_i \in N(B - b_i)$ for some $i \in \{1, 2\}$, and we may assume by symmetry that $x_2 \in N(B - b_2)$.

Subcase 2.1. $y_1, y_2, y_3 \in N(D_i'')$ for $i = 1, 2$.

We claim that there exist $\{i, j\} \subseteq \{1, 2, 3\}$ such that $G[D_1' + \{y_i, y_j\}]$ contains disjoint paths Q_1, Q_2 from x_1, y_i to v_1, y_j, respectively. This is clear if there exist y_i and y_j both with neighbors on v_1Xx_1, for X is induced, D_1 is 2-connected, and $D_1' - v_1Xx_1$ is connected (because of b_1 as $H - X$ is connected). Thus we may assume (by pigeonhole principle) that there exist y_i and y_j both with neighbors in $D_1 - v_1Xx_1$. So, since $H - X$ is connected, $G[D_1' + \{y_i, y_j\}] - v_1Xx_1$ has a path between y_i and y_j.

Without loss of generality, we may assume that $\{i, j\} = \{1, 2\}$. By (4) (with $\{i, j\} = \{1, 2\}$) and by Lemma 2.4, $G[D_2' + \{y_1, y_2, y_3\}]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in V(D_2'')$ to $S := \{b_2, u_2, x_2, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in V(P_1)$, $y_2 \in V(P_2)$, and $x_2 \in V(P_3)$.

If there exists $P \in \{P_4, P_5\}$ such that $u_2 \in V(P)$ then $P_1 \cup P_2 \cup P_3 \cup (P \cup u_2Xv_1 \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that P_4 ends at b_2 and P_5 ends at y_3.

If $b_1 \neq b_2$ then, since there is an edge between v_1Xu_2 and $B - \{b_1, b_2\}$, we see that $G[B \cup v_1Xu_2] - b_1$ contains a path Q from b_2 to v_1; hence $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup Q \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So we may assume $b_1 = b_2$. Then $G[B \cup v_Xu_2 + x_2] - b_1$ contains a path Q from v_1 to x_2, and $G[D_1 - x_1 + \{y_1, y_2\}]$ has a path R from y_1 to y_2. Hence, $P_1 \cup P_2 \cup P_3 \cup (P_5 \cup y_3x_1) \cup (Q \cup Q_1) \cup Q_2 \cup (K - y_3)$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. This completes Subcase 2.1.

So by symmetry, we may assume that $y_1, y_2 \in N(D_1'')$, $y_3 \notin N(D_1'')$, and $y_1 \in N(D_2'')$.

Subcase 2.2. $y_2 \in N(D_2'')$.

Then by (4) (with $\{i, j\} = \{1, 2\}$) and by Lemma 2.4, $G[D_2' + \{y_1, y_2, y_3\}]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in V(D_2'')$ to $S := \{b_2, u_2, x_2, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in V(P_1)$, $y_2 \in V(P_2)$, and $x_2 \in V(P_3)$. We may assume that P_4 ends in $\{b_2, u_2\}$.

First, assume that P_4 ends at u_2. If $G[D_1' + \{y_1, y_2\}] - b_1$ has disjoint paths Q_1, Q_2 from v_1, y_2 to x_1, y_1, respectively, then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup u_2Xv_1 \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that such Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D_1' + \{y_1, y_2\}] - b_1, v_1, y_2, x_1, y_1)$ is planar. So G contains TK_5 by Corollary 2.9 (as $|V(D_1'')| ge 2$ by Lemma 3.1).

Now assume P_4 ends at b_2. Let Q be a path in B from b_2 to b_1. If $G[D_1'+\{y_1,y_2\}]-v_1$ has disjoint paths Q_1, Q_2 from b_1, y_2 to x_1, y_1, respectively, then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup Q \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D_1'+\{y_1,y_2\}]-v_1, b_1, y_2, x_1, y_1)$ is planar. So G contains TK_5 by Corollary 2.9 (as $|V(D_1'')|ge2$ by Lemma 3.1).

Subcase 2.3. $y_2 \notin N(D_2'')$ and $y_2 \in N(B \cup u_2 X v_1)$.

In $G[D_1+\{y_1,y_2\}]$ we use Menger's theorem to find five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $u \in V(D_1'')$ to y_1, y_2, x_1, b_1, v_1, respectively. Since $y_2 \in N(B \cup u_2 X v_1)$, $G[B \cup u_2 X v_1 + y_2]$ has disjoint paths R_1, R_2 from $s \in \{b_1, v_1\}, y_2$ to $\{b_2, u_2\}$.

We may assume that $G[D_2' + y_1]$ contains disjoint paths L_1, L_2 from b_2, u_2 to x_2, y_1, respectively. For, otherwise, by Corollary 2.3, $(G[D_2' + y_1], b_2, u_2, x_2, y_1)$ is planar. Since in this case $\{b_2, u_2, x_2, y_1, y_3\}$ is a cut in G, G contains TK_5 by Corollary 2.9 (as $|V(D_2'')| \geq 2$ by Lemma 3.1). Similarly, we may assume that $G[D_2' + y_1]$ contains disjoint paths L_1', L_2' from b_2, u_2 to y_1, x_2, respectively.

Let $s \in V(Q_i)$ where $i \in \{4,5\}$. If $b_2 \in R_1$, then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_i \cup R_1 \cup L_1) \cup (R_2 \cup L_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume $u_2 \in V(R_1)$. Then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_i \cup R_1 \cup L_2') \cup (R_2 \cup L_1') \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Subcase 2.4. $y_2 \notin N(D_2'')$ and $y_2 \notin N(B \cup u_2 X v_1)$.

Let $v \in N(x_1) \cap V(D_1'')$ and $G' := G[D_1' + \{y_1, y_2\}]$. By Menger's theorem, $G' - x_1$ has four independent paths Q_1, Q_2, Q_3, Q_4 from v to y_1, y_2, b_1, v_1, respectively. We may assume that Q_i, $1 \leq i \leq 4$, are induced paths in G', and let $L = \bigcup_{i=1}^{5} Q_i$, where $Q_5 = v x_1$.

Note that $|N(y_2) \cap V(D_1'')| \geq 3$. So G' has an L-bridge, say J, containing an edge $y_2 u$ such that $u \notin V(Q_2 + x_1)$. We now show that L, J may be chosen so that J has an attachment in $(Q_1 \cup Q_3 \cup Q_4) - v$. For, otherwise, all attachments of J are contained in $Q_2 + x_1$. Since G is 5-connected, J has an attachment on $Q_2 - y$, say z, and we choose z so that $z Q_2 v$ is minimal. Again since G is 5-connected, there is a path in $G' - x_1$ from $y_2 Q_2 z - \{y_2, z\}$ to $(Q_1 \cup Q_3 \cup Q_4) - v$. Now letting Q_2' be obtained from Q_2 by replacing $y_2 Q_2 z$ with a path in J from y_2 to z and internally disjoint from $Q_2 + x_1$, we see that for Q_1, Q_2', Q_3, Q_4, the corresponding L, J satisfy the desired properties.

Therefore, J contains a path Y from y_2 to $y \in V(Q_1 \cup Q_3 \cup Q_4 - v)$ internally disjoint from L. Let R be a path in B between b_1 and b_2. As in Subcase 2.3, we may assume that $G[D_2' + y_1]$ contains disjoint paths L_1, L_2 from b_2, u_2 to x_2, y_1, respectively, as well as disjoint paths L_1', L_2' from b_2, u_2 to y_1, x_2, respectively.

If $y \in V(Q_1 - v)$ then $v x_1 \cup Q_2 \cup (Q_3 \cup R \cup L_1) \cup (Q_4 \cup v_1 X u_2 \cup L_2) \cup (Y \cup y Q_1 y_1) \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. If $y \in V(Q_3 - v)$ then $v x_1 \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_1 X u_2 \cup L_2') \cup (Y \cup y Q_3 b_1 \cup R \cup L_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. So we may assume $y \in V(Q_4 - v)$. Then $v x_1 \cup Q_1 \cup Q_2 \cup (Q_3 \cup R \cup L_1) \cup (Y \cup y Q_4 v_1 \cup v_1 X u_2 \cup L_2) \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. ∎

Lemma 5.2 *If $H - B$ has a 2-connected block then G contains TK_5.*

Proof. By Lemma 4.6, we may assume that no 2-connected block in H is of type I. For any 2-connected block D in $H - B$, recall the notation D'', D', b_D, u_D, v_D. Since G is 5-connected, $|N(D'') \cap \{y_1, y_2, y_3\}| \geq 2$. So $|V(D'')| \geq 2$ by Lemma 3.1.

Case 1. $|N(D'') \cap \{y_1, y_2, y_3\}| = 2$ for each 2-connected block D in $H - B$.

Let D be a 2-connected block of $H - B$. Without loss of generality, let $y_1, y_2 \in N(D'')$ and $y_3 \notin N(D'')$. Using Menger's theorem, we find independent paths P_1, P_2, P_3, P_4, P_5 in $G[D' + \{y_1, y_2\}]$ from some vertex $u \in V(D'')$ to y_1, y_2, u_D, v_D, b_D, respectively.

If $y_1, y_2 \in N(B)$ then in $G[B + \{y_1, y_2\}]$ we find a path Q from y_1 to y_2; so $P_1 \cup P_2 \cup (P_3 \cup u_D X x_1) \cup (P_4 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume $y_1 \notin N(B)$; hence by Lemma 5.1 we may assume $y_2, y_3 \in N(B)$.

Subcase 1.1. $N(y_1) \not\subseteq V(D) \cup \{x_1, x_2\}$.

Then $G - \{y_2, y_3\}$ contains a path P from y_1 to some vertex $u \in V(B \cup X) - (V(D') \cup \{x_1, x_2\})$ and internally disjoint from $B \cup D' \cup X$. If $u \in V(B)$ then $G[B \cup P + y_2]$ has a path Q between y_1 and y_2, and $P_1 \cup P_2 \cup (P_3 \cup u_D X x_1) \cup (P_4 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So we may assume that $u \notin V(B)$ for any choice of P. Hence, since $H - X$ is connected, all neighbors of y_1 outside D are on X; in particular, $u \in V(u_D X x_1 - \{u_D, x_1\}) \cup V(v_D X x_2 - \{v_D, x_2\})$ and $V(P) = \{y_1, u\}$. By symmetry we may assume that $u \in V(u_D X x_1) - \{u_D, x_1\}$. Since X is induced and $H - X$ is connected and by Lemma 3.1, H contains a path from u to B and internally disjoint from $B \cup D \cup X$, which can be extended through $G[B + y_2]$ to a path R from u to y_2. If $G[D' + \{y_1, y_2\}] - b_D$ has disjoint paths R_1, R_2 from y_1, u_D to y_2, v_D, respectively, then $uy_1 \cup R \cup uXx_1 \cup (uXu_D \cup R_2 \cup v_D X x_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume that such R_1, R_2 do not exist. So by Corollary 2.3, $(G[D' + \{y_1, y_2\}] - b_D, y_1, u_D, y_2, v_D)$ is planar. Now G contains TK_5 by Corollary 2.9.

Subcase 1.2. $N(y_1) \subseteq V(D) \cup \{x_1, x_2\}$, and $N(y_2) \subseteq V(D') \cup \{x_1, x_2\}$.

Then $N(y_2) \cap V(B) = \{b_D\}$, and $\{b_D, u_D, v_D, x_1, x_2\}$ is a cut in G separating $B + y_3$ from $D' + \{y_1, y_2\}$. So $x_1 \neq u_D$ and $x_2 \neq v_D$, as G is 5-connected. Therefore, $H - D$ contains a path X' from x_1 to x_2. Note that D is 2-connected; so it is contained in a 2-connected block of $H - X'$. Also note that y_1 and y_2 each have at least two neighbors in D. So it follows from Lemma 3.2 and the choice of X that y_2, y_3 should each have at least two neighbors in B, a contradiction as we are in Subcase 1.2.

Subcase 1.3. $N(y_1) \subseteq V(D) \cup \{x_1, x_2\}$, and $y_2 \in N(F'')$ for some 2-connected block F in $H - B$.

Let $v \in N(y_2) \cap V(F'')$. Without loss of generality, assume that $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order. Since $N(y_1) \subseteq V(D) \cup \{x_1, x_2\}$, $y_1 \notin N(F'')$; and since G is 5-connected, $y_3 \in N(F'')$. Let Q be a path in $G[B+y_3]$ from y_3 to b_D. If $G[F' + \{y_2, y_3\}] - b_F$ contains disjoint paths Q_1, Q_2 from u_F, y_2 to v_F, y_3, respectively, then $P_2 \cup (P_5 \cup Q) \cup (P_3 \cup u_D X v_F \cup Q_1 \cup u_F X x_1) \cup (P_4 \cup v_D X x_2) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume that such Q_1, Q_2 do not exist. Then by Corollary 2.3, $G[F' + \{y_2, y_3\}] - b_F, u_F, y_2, v_F, y_3)$ is planar. Hence G contains TK_5 by Corollary 2.9 (as $|V(F')| \geq 7$ by Lemma 3.1).

Subcase 1.4. $N(y_1) \subseteq V(D) \cup \{x_1, x_2\}$, $N(y_2) \not\subseteq V(D') \cup \{x_1, x_2\}$, and $y_2 \notin N(F'')$ for any 2-connected block F in $H - B$ other than D.

Then, since G is 5-connected, D is the unique 2-connected block of $H - B$. So let $v \in N(y_2)$ such that $v \in V(B - b_D) \cup (V(X) - V(u_D X v_D + \{x_1, x_2\}))$. By symmetry, we may assume that $v \in V(B - b_D) \cup V(x_1 X u_D - \{x_1, u_D\})$.

We may further assume that $v \in V(B - b_D)$. For, otherwise, $N(y_2) \cap V(B) = \{b_D\}$ and we may assume $v \in V(x_1 X u_D) - \{x_1, u_D\}$. Hence by Lemma 3.1, $y_3 \in N(B - b_D)$. Thus, $G[B + \{v, y_3\}]$ contains independent paths R_1, R_2 from b_D to y_3, v, respectively. Now $y_2 b_D \cup R_2 \cup v y_2 \cup (x_1 y_2 \cup x_1 X v \cup x_1 y_3 \cup R_1) \cup (P_2 \cup P_5 \cup P_1 \cup y_1 x_1 \cup P_3 \cup u_D X v)$ is a TK_5 in G with branch vertices b_D, u, v, x_1, y_2.

We may assume that $G[D' + y_2]$ contains disjoint paths Q_1, Q_2 from b_D, v_D to y_2, u_D, respectively; for otherwise by Corollary 2.3, $(G[D' + y_2], b_D, v_D, y_2, u_D)$ is planar, and so G contains TK_5 by Corollary 2.9. Similarly, we may assume that $G[D' + y_2]$ contains disjoint paths Q_1', Q_2' from b_D, v_D to u_D, y_2, respectively, as well as disjoint paths Q_1'', Q_2'' from b_D, u_D to v_D, y_2, respectively.

Suppose y_3 has at least two neighbors in B. Then $G[B + y_3]$ contains independent paths R_1, R_2 from y_3 to v, b_D, respectively. Then $P_2 \cup (P_5 \cup R_2) \cup (P_3 \cup u_D X x_1) \cup (P_4 \cup v_D X x_2) \cup (R_1 \cup v y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Thus we may assume that y_3 has only one neighbor in B. Therefore y_3 must have at least two neighbors in $(u_D X x_1 - x_1) \cup (v_D X x_2 - x_2)$. By symmetry, we may assume that y_3 has a neighbor in $v_D X x_2 - x_2$.

First, assume that y_3 has two neighbors $w_1, w_2 \in V(v_D X x_2 - x_2)$, with $w_1 \in V(x_2 X w_2)$. Since $v \in V(B - b_D)$ and $|N(w_1) \cap V(B)| \geq 2$ (by Lemma 3.1 and 5-connectedness of G), $G[B + \{w_1, y_2\}]$ has independent paths R_1, R_2 from w_1 to b_D, y_2, respectively. So $w_1 X x_2 \cup (R_1 \cup Q_1' \cup u_D X x_1) \cup R_2 \cup w_1 y_3 \cup (y_3 w_2 \cup w_2 X v_D \cup Q_2') \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_2, y_3.

Next assume that y_3 has exactly one neighbor $w_1 \in V(v_D X x_2 - x_2)$. Then y_3 also has a neighbor $w_2 \in V(u_D X x_1 - x_1)$. Clearly, $x_1, x_2 \in N(B)$; so $G[B + \{x_1, x_2\}]$ contains a path X' between x_1 and x_2. We claim that $|N(y_2) \cap V(B)| \geq 2$; otherwise, we have a contradiction to the choice of X and Lemma 3.2 because D is in a 2-connected block of $H - X'$, $y_1, y_2 \in N(D'')$, and $|N(y_1) \cap V(D)| \geq 3$. Thus y_2 has a neighbor $w \in V(B)$ such that $x_1 \in N(B - w)$. Suppose $w_1 \neq v_D$. In $G[D' + \{y_1, y_2\}] - \{b_D, u_D\}$ we find a path Q from v_D to y_2 through y_1, which exists because D is 2-connected and $N(y_1) \subseteq V(D') \cup \{x_1, x_2\}$. In $G[B \cup u_D X x_1 + w_1]$ we find independent paths R_1, R_2 from w_1 to x_1, w, respectively. Then $R_1 \cup w_1 X x_2 \cup (R_2 \cup w y_2) \cup w_1 X v_D \cup Q \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_1, y_2. So we may assume $w_1 = v_D$. In $G[D + y_2]$ we find independent paths R_1, R_2 from w_1 to u_D, y_2, respectively, and let R be a path in $G[B + \{y_2, y_3\}]$ from y_2 to y_3. Now $w_1 y_3 \cup R_2 \cup w_1 X x_2 \cup (R_1 \cup u_D X x_1) \cup R \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_2, y_3.

Case 2. There exists a 2-connected block D in $H - B$ such that $\{y_1, y_2, y_3\} \subseteq N(D'')$.

By Lemma 5.1, we may assume that $y_1, y_2 \in N(B)$. Let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2.

We may further assume that $G - \{y_1, y_2\}$ contains no path from y_3 to B and internally disjoint from $B \cup D' \cup X$. For, let P be such a path in H. Then, for any $\{s, t\} \subseteq \{1, 2, 3\}$, $G[B \cup P + \{y_s, y_t\}]$ contains a path Q_{st} between y_s and y_t. Note that D contains independent paths from some $u \in V(D'')$ to u_D, v_D, respectively. So by Lemma 2.4, $G[D' + \{y_1, y_2, y_3\}]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from u to $S := \{b_D, u_D, v_D, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $u_D \in P_1$, and $v_D \in P_2$. Without loss of generality, we may assume that P_3 ends at y_i and P_4 ends at y_j. Now $(P_1 \cup u_D X x_1) \cup (P_2 \cup v_D X x_2) \cup P_3 \cup P_4 \cup Q_{ij} \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_j.

In particular, we may assume $y_3 \notin N(B)$.

Subcase 2.1. $D - u_D$ is not 2-connected or $D - v_D$ is not 2-connected.

By symmetry we may assume that $D - u_D$ is not 2-connected. Let C denote an endblock of $D - u_D$, and let v be the cut vertex of $D - u_D$ such that $v \in V(C)$ and $v_D \notin V(C - v)$. By Lemma 3.4, we may assume $b_D \in N(C - v)$. By Lemma 3.5 we may assume that $v_D \neq v$. Since G is 5-connected, $|N(C - v) \cap \{y_1, y_2, y_3\}| \geq 2$; hence by Lemma 3.1, C is 2-connected.

Since D is 2-connected, $D - C$ has a path P from u_D to v_D. So C is contained in a 2-connected block of $H - (x_1 X u_D \cup P \cup v_D X x_2)$. Hence, $|N(C - v) \cap \{y_1, y_2, y_3\}| = 2$, for, otherwise, it follows from Lemma 3.2 and the choice of X that $\{y_1, y_2, y_3\} \subseteq N(B)$, contradicting the assumption that $y_3 \notin N(B)$.

Suppose $y_1, y_2 \in N(C - v)$. Then $y_3 \notin N(C - v)$. Since G is 5-connected, there are five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 in $G[C + \{b_D, u_D, y_1, y_2\}]$ from some vertex $u \in V(C - v)$ to u_D, v, y_1, y_2, b_D, respectively. Let R denote a path in $D - u_D - (C - v)$ from v to v_D. Then $(Q_1 \cup u_D X x_1) \cup (Q_2 \cup R \cup v_D X x_2) \cup Q_3 \cup Q_4 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Thus, by symmetry, we may assume that $y_2, y_3 \in N(C - v)$. So $y_1 \notin N(C - v)$. Let $C' := (D - u_D) - (C - v)$.

We may assume that $G[C' + \{u_D, y_1\}] - v$ has three independent paths from some vertex $u \in V(C') - \{v, v_D\}$ to u_D, v_D, y_1, respectively. For, suppose not. Then v is a cut vertex of C' separating v_D from $N(y_1) \cap V(C')$. Let C_v denote the v-bridge of C' containing v_D, and let C_y be a v-bridge of C' such that $y_1 \in N(C_y - v)$. Let X' be a path obtained from X by replacing $u_D X v_D$ with a path in $G[C_v + u_D] - v$ from u_D to v_D. Then $X' \cap (B \cup C \cup C_y) = \emptyset$. Suppose $y_3 \in N(C_y - v)$. Then $G[C_y + \{y_1, y_3\}] - v$ has a path Q_1 between y_1 and y_3. Let Q_2 be path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2, and Q_3 be a path in $G[C + \{y_2, y_3\}] - v$ between y_2 and y_3. Now $Q_1 \cup Q_2 \cup Q_3 \cup X' \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. Thus we may assume assume that $y_3 \notin N(C_y - v)$. Hence, since G is 5-connected, $b_D, u_D, y_1, y_2 \in N(C_y - v)$. So by Menger's theorem, $G[C_y + \{b_D, u_D, y_1, y_2\}]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some vertex $u \in V(C_y - v)$ to u_D, v, y_1, y_2, b_D, respectively. Note that the path Q_2 can be extended through C_v to a path Q_2' ending at v_D. Then $(Q_1 \cup u_D X x_1) \cup (Q_2' \cup v_D X x_2) \cup Q_3 \cup Q_4 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Let $S = \{b_D, u_D, v_D, v, y_1\} \cup (N(C') \cap \{y_2, y_3\})$. So $G[C' + S]$ is $(5, S)$-connected. Hence, by Lemma 2.4, $G[C' + S]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to S such that $V(Q_i \cap Q_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(Q_i) \cap S| = 1$ for $1 \leq i \leq 5$, $u_D \in V(Q_1)$, $v_D \in V(Q_2)$, and $y_1 \in V(Q_3)$. We may assume Q_4 ends in $\{v, y_2, y_3\}$.

If $y_2 \in V(Q_4)$ then $Q_3 \cup Q_4 \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. If $v \in V(Q_4)$ then we extend Q_4 through $G[C + y_2]$ to a path Q_4' ending at y_2; now $Q_3 \cup Q_4' \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume that $y_3 \in V(Q_4)$. Let Q' be a path in $G[B \cup C + \{y_1, y_3\}] - v$ between y_1 and y_3 (using $b_D \in N(C - v)$); then $Q_3 \cup Q_4 \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q' \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_3.

Subcase 2.2. $D - u_D$ and $D - v_D$ are 2-connected.

First, assume $u_D = x_1$ and $v_D = x_2$. Then since $y_3 \notin N(B)$, $\{b_D, x_1, x_2, y_1, y_2\}$ is a cut in G separating B from D. In $G[B + \{x_1, x_2, y_1, y_2\}]$ we use Menger's theorem to find five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in V(B - b_D)$ to x_1, x_2, y_1, y_2, b_D, respectively. In $G[D'' + \{y_1, y_2\}]$ we find a path Q' between y_1 and y_2. Now $P_1 \cup P_2 \cup P_3 \cup P_4 \cup Q' \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Thus we may assume that $u_D \neq x_1$. We may further assume that $v_D = x_2$, and H contains no path from x_2 to B and internally disjoint from $B \cup D' \cup X$. For, otherwise, since $u_D \neq x_1$, H contains a path X' from x_1 to x_2 and internally disjoint from $D \cup X$. Thus $D - v_D$ is contained in a 2-connected block of $H - X'$. Since $y_1, y_2, y_3 \in N(D'')$ and $y_3 \notin N(B)$, it follows from Lemma 3.2 and the choice of X that G contains TK_5.

Suppose $N(y_3) \subseteq V(D)$. Then $\{b_D, u_D, x_1, y_1, y_2\}$ is a cut in G separating $B \cup u_D X x_1$ from D'. Let G_1 denote the $\{b_D, u_D, x_1, y_1, y_2\}$-bridge of G containing $B \cup u_D X x_1$. Since $D - u_D$ is 2-connected, $G[D'' + \{v_D, y_1, y_2\}]$ has independent paths from some $u \in V(D'')$ to y_1, y_2, v_D, respectively. So in $G[D' + \{y_1, y_2, y_3\}]$ we use Lemma 2.4 to find five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to $S := \{b_D, u_D, v_D, y_1, y_2, y_3\}$ such that $V(Q_i \cap Q_j) = \{u\}$ for $1 \leq i < j \leq 5$, $|V(Q_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in V(Q_1)$, $y_2 \in V(Q_2)$, and $v_D \in V(Q_3)$. We may assume Q_4 ends in $\{b_D, u_D\}$. If $u_D \in V(Q_4)$ then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_4 \cup u_D X x_1) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume $b_D \in V(Q_4)$. If $G_1 - u_D$ contains disjoint paths R_1, R_2 from x_1, y_2 to b_D, y_1, respectively, then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_4 \cup R_1) \cup R_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that such R_1, R_2 do not exist; then by Corollary 2.3, $(G_1 - u_D, x_1, y_2, b_D, y_1)$ is planar. Hence G contains TK_5 by Corollary 2.9 (as $|V(G_1)| \geq 7$ by Lemma 3.1).

Thus, we may assume that there exists $u \in N(y_3) \cap V(u_D X x_1 - \{u_D, x_1\})$.

We claim that for any permutation ijk of $\{1, 2, 3\}$ there are (not necessarily distinct) vertices v_1, v, v_2 on X in order from x_1 to u_D or there exist a 2-connected block $F \neq D$ in $H - B$ and $v \in F''$, with $v_1 = u_F$ and $v_2 = v_F$, such that $H + \{y_i, y_j\}$ has independent paths P_1, P_2, P_3, P_4 from v to v_1, v_2, y_i, y_j, respectively, and internally disjoint from $v_1 X x_1 \cup v_2 X x_2 \cup D \cup K$. This is easy to verify when $u \notin V(F)$ for any 2-connected block F of $H - B$; as in this case u has at least two neighbors in B (by Lemma 3.1) and, since $y_1, y_2 \in N(B)$, we get the desired paths by setting $v = v_1 = v_2 = u$, letting $P_1 = \{v_1\}$ and $P_2 = \{v_2\}$, and finding independent paths P_3, P_4 in $G[B + \{v, y_1, y_2\}]$ from v to y_1, y_2, respectively. So we may assume that $u \in V(F)$ for some 2-connected block F in $H - B$. Since $H - X$ is connected and X is induced, F contains a path R from u to b_F and internally disjoint from X. So, because $y_1, y_2 \in N(B)$, the claim holds whenever $3 \in \{i, j\}$ by setting $v_1 = v_2 = v = u$ and letting $P_1 = \{v_1\}$, $P_2 = \{v_2\}$, $P_4 = uy_3$, and P_3 be the union of R and a path in $G[B + y_1]$ from b_F to y_1. Now suppose $\{i, j\} = \{1, 2\}$. Let $v_1 = u_F$ and $v_2 = v_F$. First, assume $y_i \in N(F'')$ and $y_j \notin N(F'')$. Then by Menger's theorem we find five independent paths $P_1, P_2, P_3, P_4', P_5'$ in $G[F + \{y_i, y_3\}]$ from some vertex $v \in V(F'')$ to v_1, v_2, y_i, b_F, y_3, respectively. By extending P_4' through $G[B + y_j]$ to a path P_4 ending at y_j, we find the desired paths. So we may assume that $y_i, y_j \in N(F'')$. Note that $G[F + y_i]$ contains independent paths from some vertex $v \in F''$ to v_1, v_2, y_i, respectively (as F is 2-connected). So by Lemma 2.4, $G[F' + \{y_1, y_2, y_3\}]$ contains five independent paths $P_1, P_2, P_3, P_4', P_5'$ from v to $S := \{b_F, v_1, v_2, y_i, y_j, y_3\}$, such that $V(P_s \cap P_t) = \{v\}$ for $1 \leq s < t \leq 5$, $|V(P_s) \cap S| = 1$ for $1 \leq s \leq 5$, $v_1 \in V(P_1)$, $v_2 \in V(P_2)$, and $y_i \in V(P_3)$. We may assume P_4' ends in $\{b_F, y_j\}$. If P_4' ends at y_j then let $P_4 := P_4'$; if P_4' ends at b_F then we extend P_4' through $G[B + y_j]$ to a path P_4 ending at y_j. Now P_1, P_2, P_3, P_4 give the desired paths.

Let D^* be obtained from $G[D + \{y_1, y_2, y_3\}]$ by identifying y_1 and y_2, and use y to denote the new vertex. Recall that $v_D = x_2$.

Suppose D^* contains disjoint paths Q_1, Q_2 from u_D, y to v_D, y_3, respectively. Then in G, Q_2 is a path from y_i to y_3 for some $i \in \{1,2\}$. Using the paths P_1, P_2, P_3, P_4 for $\{i,j\} = \{i,3\}$, we see that $(P_1 \cup v_1Xx_1) \cup (P_2 \cup v_2Xu_D \cup Q_1) \cup P_3 \cup P_4 \cup Q_2 \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_i, y_3.

Thus we may assume that such Q_1, Q_2 do not exist. So by Lemma 2.3, (D^*, u_D, y, v_D, y_3) is 3-planar. Since D is 2-connected, we see that $G[D + \{y_1, y_2\}]$ has disjoint paths R_1, R_2 from u_D, y_2 to v_D, y_1, respectively. Therefore, using the paths P_1, P_2, P_3, P_4 for $\{i,j\} = \{1,2\}$, we see that $(P_1 \cup v_1Xx_1) \cup (P_2 \cup v_2Xu_D \cup R_1) \cup P_3 \cup P_4 \cup R_2 \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. ∎

6 $H - B = X$

By Lemmas 4.6 and 5.2, it suffices to deal with the case when $H - B = X$ is simply an induced path. First, we show that two of $\{y_1, y_2, y_3\}$ each have at least two neighbors in B.

Lemma 6.1 *Suppose $H - B = X$. Then G contains TK_5, or $|\{y_i : i \in \{1,2,3\}$ and $|N(y_i) \cap V(B)| \geq 2\}| \geq 2$.*

Proof. Suppose on the contrary that $|\{y_i : i \in \{1,2,3\}$ and $|N(y_i) \cap V(B)| \geq 2\}| \leq 1$. Then since G is 5-connected and X is induced in G, there exist distinct vertices $v_1, v_2 \in X - \{x_1, x_2\}$ such that each v_i is a neighbor of some y_j and y_j has at most one neighbor in B. We choose v_1 and v_2 so that v_1Xv_2 is maximal.

Without loss of generality, we may assume that x_1, v_1, v_2, x_2 occur on X in this order, $|N(y_i) \cap V(B)| \leq 1$ for $i = 1,2$, and $v_1 \in N(y_1)$ and $v_2 \in N(\{y_1, y_2\})$. Note that, since G is 5-connected and by Lemma 3.1, each v_i has at least two neighbors in B.

First, assume that $v_2 \in N(y_1)$. Without loss of generality, let $w_2, u_2 \in N(y_2) \cap V(X - \{x_1, x_2\})$ such that v_1, w_2, u_2, v_2 occur on X in order. In $G[B + \{v_1, x_2\}]$ there is a path P from v_1 to x_2. Thus $v_1Xx_1 \cup P \cup v_1y_1 \cup (v_1Xw_2 \cup w_2y_2) \cup (y_2u_2 \cup u_2Xv_2 \cup v_2y_1) \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_2.

Hence we may assume that $v_2 \in N(y_2)$. For $i = 1,2$, let $w_i \in N(y_i) \cap V(v_1Xv_2 - \{v_1, v_2\})$. Note that the only possible cut vertex in $G[B + \{v_1, v_2, x_1\}]$ exists when x_1 has a unique neighbor in B. Thus $G[B + \{v_1, v_2, x_1\}]$ has independent paths P, Q from v_2 to x_1, v_1, respectively. Then $P \cup v_2Xx_2 \cup v_2y_2 \cup (Q \cup v_1y_1) \cup (y_1w_1 \cup w_1Xw_2 \cup w_2y_2) \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_1, y_2. ∎

We now reduce the problem to the case when $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1,2,3$. We will make use of Lemma 2.5.

Lemma 6.2 *Suppose $H - B = X$. Then G contains TK_5, or $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1,2,3$.*

Proof. By Lemma 6.1, we may assume that $|N(y_i) \cap V(B)| \geq 2$ for $i = 1,2$.

Suppose there exists some $i \in \{1,2,3\}$ such that $y_i \in N(B)$ and $y_i \in N(X - \{x_1, x_2\})$. Let $u \in N(y_i) \cap V(X - \{x_1, x_2\})$. Then there exists $j \in \{1,2\} - \{i\}$ such that $G[B + \{u, y_i, y_j\}]$ contains two independent paths P_1 and P_2 from y_j to u, y_i respectively. Now $uy_i \cup P_1 \cup X \cup P_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_j.

Thus, we may assume that $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2$, and $N(y_3) \subseteq V(X)$ or $N(y_3) \subseteq V(B) \cup \{x_1, x_2\}$. We may further assume that $N(y_3) \subseteq V(X)$, or else the assertion of the lemma holds.

For any $u \in V(X) - \{x_1, x_2\}$, since X is induced and by Lemma 3.1, $|N(u) \cap V(B)| \geq 2$. So $R_u = G[B + \{u, y_1, y_2\}]$ is 2-connected. If R_u contains a cycle T containing $\{u, y_1, y_2\}$ then $T \cup X \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume that such T does not exist. Then, by Lemma 2.5, R_u has a 2-cut S separating y_1 from $\{u, y_2\}$. Clearly, $S \subseteq V(B)$ (as B is 2-connected) and in B, S separates $N(y_1)$ from $N(y_2)$. We choose u and S such that the component C of $R_u - S$ containing y_1 is minimal. Since G is 5-connected and $N(y_3) \subseteq V(X)$, there exist $v \in V(X) - \{x_1, x_2, u\}$ such that $v \in N(C)$. Now the minimality of C implies that no 2-cut in $R_v := G[B + \{v, y_1, y_2\}]$ separates y_1 from $\{v, y_2\}$. Hence, by Lemma 2.5, R_v has a cycle T' containing $\{v, y_1, y_2\}$. Now $T' \cup X \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. ∎

We now show that G contains TK_5. By Lemma 6.2, we may assume that $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2, 3$; so $R := G[B + \{y_1, y_2, y_3\}]$ is 2-connected and each y_i has degree at least 3 in R.

If R has a cycle C containing $\{y_1, y_2, y_3\}$, then $C \cup X \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. So we may assume that such a cycle does not exist in R. Then by Lemma 2.5, we have three cases to consider.

Case 1. There exists a 2-cut S in R and there exist three distinct components D_1, D_2, D_3 of $R - S$ such that $y_i \in V(D_i)$ for each $i \in \{1, 2, 3\}$.

Let $S = \{a, b\}$. Since each y_i has degree at least 3 in R, $|V(D_i - y_i)| \geq 1$ for $1 \leq i \leq 3$. Since G is 5-connected, $N(D_i - y_i) \cap V(X - \{x_1, x_2\}) \neq \emptyset$. Moreover, since B is 2-connected, $G[D_i + S] - y_i$ is a chain of blocks from a to b; so let $Q_i \subseteq G[D_i \cup S]$ be a path from a to b containing y_i.

We may assume $ab \notin E(G)$. For, suppose $ab \in E(G)$. Since X is induced, x_1 has at least two neighbors outside $D_i \cup D_j$ for some $\{i, j\} \subseteq \{1, 2, 3\}$. Then $G[R - (D_i \cup D_j) + x_1]$ has independent paths L_1, L_2 from x_1 to a, b, respectively. Now $Q_i \cup Q_j \cup ab \cup y_i x_2 y_j \cup L_1 \cup L_2 \cup x_1 y_i \cup x_1 y_j$ is a TK_5 in G with branch vertices a, b, x_1, y_i, y_j.

Let A_i be a path in G from a to some $a_i \in N(D_i - y_i) \cap V(X - \{x_1, x_2\})$ which is internally disjoint from $(R - D_i) \cup X$. We may assume $|\{a_1, a_2, a_3\}| \geq 2$. For, suppose $a_1 = a_2 = a_3$. Then by symmetry, we may assume that $G[R + a_1]$ has independent paths P_i (for $i = 1, 2$) from a_1 to $q_i \in V(y_i Q_i b)$ and internally disjoint from $Q_i \cup D_3$. Now $a_1 X x_1 \cup a_1 X x_2 \cup (P_1 \cup q_1 Q_1 y_1) \cup (P_2 \cup q_2 Q_2 y_2) \cup (y_1 Q_1 a \cup a Q_2 y_2) \cup K$ is a TK_5 in G with branch vertices a_1, x_1, x_2, y_1, y_2.

We may further assume that $R - S$ has only three components and $N(a) \cap V(X) = \emptyset$. Otherwise, there exists a path A from a to some $a' \in V(X)$ which is internally disjoint from $D_1 \cup D_2 \cup D_3 \cup X$. Without loss of generality, we may assume that $a' \in V(x_1 X a_3 - a_3)$ (since $|\{a_1, a_2, a_3\}| \geq 2$). Then $a Q_1 y_1 \cup a Q_2 y_2 \cup (y_1 Q_1 b \cup b Q_2 y_2) \cup (A_3 \cup a_3 X x_2) \cup (A \cup a' X x_1) \cup K$ is a TK_5 in G with branch vertices a, x_1, x_2, y_1, y_2.

Therefore, a has degree at least 5 in R. By Lemma 3.1, we may assume $|N(a) \cap \{y_1, y_2, y_3\}| \leq 1$. Hence, since $ab \notin E(G)$, there exists some $i \in \{1, 2, 3\}$ such that $|(N(a) \cap V(D_i)) - y_i| \geq 2$, say $i = 1$.

We claim that $G[D_1 \cup X + a] - y_1$ has independent paths P_1, P_2 from a to distinct $c_1, c_2 \in V(X)$ and internally disjoint from X. For, suppose P_1, P_2 do not exist. Then $G[D_1 \cup X + a] - y_1$

has a cut vertex c separating a from X. Hence, $\{a, b, c, y_1\}$ is a cut in G as $|(N(a) \cap V(D_1)) - y_1| \geq 2$, a contradiction.

Without loss of generality, we may assume that x_1, c_1, c_2, x_2 occur on X in order. Then $(P_1 \cup c_1Xx_1) \cup (P_2 \cup c_2Xx_2) \cup aQ_2y_2 \cup aQ_3y_3 \cup (y_2Q_2b \cup bQ_3y_3) \cup K$ is a TK_5 in G with branch vertices a, x_1, x_2, y_2, y_3.

Case 2. There exist a vertex b of R, 2-cuts S_1, S_2, S_3 in R, and components D_i of $R - S_i$ containing y_i, for all $i \in \{1, 2, 3\}$, such that $S_1 \cap S_2 \cap S_3 = \{b\}$, $S_i - \{b\} = \{a_i\}$ where a_1, a_2, a_3 are distinct, and D_1, D_2, D_3 are pairwise disjoint.

For convenience, let $R' := R - (D_1 \cup D_2 \cup D_3)$. We choose S_1, S_2, S_3 such that $D_1 \cup D_2 \cup D_3$ is maximal. Then $R' - b$ is connected.

As in Case 1, let $Q_i \subseteq G[D_i \cup S_i]$ be a path from a_i to b which passes through y_i, and let A_i be a path from a_i to $c_i \in N(D_i - y_i) \cap V(X - \{x_1, x_2\})$ and internally disjoint from $(R - D_i) \cup X$. We may choose c_i so that $|\{c_1, c_2, c_3\}| \geq 2$; the proof is the same as in Case 1 (for showing $|\{a_1, a_2, a_3\}| \geq 2$) since $R' - b$ is connected.

Suppose there exists a vertex $u \in V(R') - \{a_1, a_2, a_3, b\}$ such that $R' - b$ has two independent paths from u to two distinct vertices of $\{a_1, a_2, a_3\}$, say a_1 and a_2, and a_3 is not in these paths. Let $S = \{a_1, a_2, a_3, b\} \cup (N(R') \cap V(X))$. Note that $G[R' + S] - b$ is $(4, S - \{b\})$-connected and $R' - a_3$ contains independent paths from u to a_1, a_2, respectively. So by Lemma 2.4, there exist four independent paths P_1, P_2, P_3, P_4 in $G[R' + S] - b$ from u to $S - \{b\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i < j \leq 4$, $|V(P_i) \cap (S - \{b\})| = 1$ for $1 \leq i \leq 4$, $a_1 \in V(P_1)$, and $a_2 \in V(P_2)$. We may assume that P_3 ends at some vertex $v \in V(X)$ and P_4 ends at some vertex $w \in V(X) \cup \{a_3\}$. If $w \in V(X)$ then by symmetry we may assume $v \in x_1Xw$; now $(P_1 \cup a_1Q_1y_1) \cup (P_2 \cup a_2Q_2y_2) \cup (P_3 \cup vXx_1) \cup (P_4 \cup wXx_2) \cup (y_1Q_1b \cup bQ_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that $w = a_3$. If $v \neq c_3$ then by symmetry we may assume $v \in x_1Xc_3$; now $(P_1 \cup a_1Q_1y_1) \cup (P_2 \cup a_2Q_2y_2) \cup (P_3 \cup vXx_1) \cup (P_4 \cup A_3 \cup c_3Xx_2) \cup (y_1Q_1b \cup bQ_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that $v = c_3$. Then $v \neq c_1$ or $v \neq c_2$. By symmetry, we may assume that $v \neq c_2$ and $v \in x_1Xc_2$. Then $(P_1 \cup a_1Q_1y_1) \cup (P_4 \cup a_3Q_3y_3) \cup (P_3 \cup vXx_1) \cup (P_2 \cup A_2 \cup c_2Xx_2) \cup (y_1Q_1b \cup bQ_3y_3)$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_3.

So we may assume that for any vertex $u \in V(R') - \{a_1, a_2, a_3, b\}$, there exists a 2-cut $S_u = \{b, b_u\}$ in R' separating u from $\{a_1, a_2, a_3\}$. We choose u and S_u so that the S_u-bridge of R' containing u is maximal. Then $b_u \in \{a_1, a_2, a_3\}$, say $b_u = a_3$, and $R' - \{a_1, a_2\}$ is the unique b_u-bridge of $R' - b$ containing u. Since $R - \{y_1, y_2, y_3\}$ is 2-connected, $R[\{a_1, a_2, a_3\}]$ must be connected.

We may assume that $R[\{a_1, a_2, a_3\}]$ is a triangle. Otherwise, for some permutation ijk of $\{1, 2, 3\}$, we have $a_ia_j \notin E(G)$ and $a_ia_k, a_ja_k \in E(G)$. Then $\{b, a_k\}$ is a 2-cut of R such that y_1, y_2, y_3 belong to three different components of $R - \{b, a_k\}$ whose union properly contains $D_1 \cup D_2 \cup D_3$, contradicting the choice of S_1, S_2, S_3 (to maximize $D_1 \cup D_2 \cup D_3$).

Suppose for some $i \in \{1, 2\}$, $N(a_i) \not\subseteq \{a_1, a_2, a_3, b\} \cup V(D_i)$. Then H has an edge a_iv_i with $v_i \in V(X)$. Since $\{a_i, b, y_i, v_i\}$ is not a cut in G, we see that A_i may be chosen so that $c_i \neq v_i$. Without loss of generality, we may assume that $v_i \in V(x_1Xc_i - c_i)$. Let $\{i, j\} = \{1, 2\}$. Now $(A_i \cup c_iXx_2) \cup (a_iv_i \cup v_iXx_1) \cup (a_ia_j \cup a_jQ_jy_j) \cup (a_ia_3 \cup a_3Q_3y_3) \cup (y_jQ_jb \cup bQ_3y_3) \cup K$ is a TK_5 in G with branch vertices a_i, x_1, x_2, y_j, y_3.

Thus we may assume that for all $i \in \{1, 2\}$, $N(a_i) \subseteq \{a_1, a_2, a_3, b\} \cup V(D_i)$. Suppose $|N(a_1) \cap V(D_1 - y_1)| \geq 2$. Then, since G is 5-connected, there exist two independent paths

P_1, P_2 in $G[(D_1 + a_1) \cup X] - y_1$ from a_1 to $c_1, c_2 \in V(X)$ respectively, and internally disjoint from X. Without loss of generality, assume $c_1 \in V(x_1 X c_2)$. Now $(P_1 \cup c_1 X x_1) \cup (P_2 \cup c_2 X x_2) \cup (a_1 a_3 \cup a_3 Q_3 y_3) \cup (a_1 a_2 \cup a_2 Q_2 y_2) \cup (y_3 Q_3 b \cup b Q_2 y_2) \cup K$ is a TK_5 in G with branch vertices a_1, x_1, x_2, y_2, y_3.

Thus, we may assume that $|N(a_1) \cap V(D_1 - y_1)| \leq 1$. Hence, since G is 5-connected, $y_1 a_1 \in E(G)$. Let $y \in V(D_1)$ be a neighbor of y_1. Then D_1 has two independent paths from y to a_1, b, respectively. So by Lemma 2.4, $G[D_1 + \{a_1, b\} + N(D_1) \cap V(X)] - y_1$ has four independent paths P_1, P_2, P_3, P_4 from y to $\{a_1, b\} \cup (N(D_1) \cap V(X))$ such that $V(P_i \cap P_j) = \{y\}$ for $1 \leq i < j \leq 4$, $a_1 \in V(P_1)$ and $b \in V(P_2)$. Let $v_1, v_2 \in V(X)$ with $v_1 \in V(P_3)$ and $v_2 \in V(P_4)$, and assume that x_1, v_1, v_2, x_2 occur on X in order. Now $y y_1 \cup (P_2 \cup b Q_2 y_2) \cup (P_3 \cup v_1 X x_1) \cup (P_4 \cup v_2 X x_2) \cup (y_1 a_1 a_2 \cup a_2 Q_2 y_2) \cup K$ is a TK_5 in G with branch vertices y, x_1, x_2, y_1, y_2.

Case 3. There exist pairwise disjoint 2-cuts S_1, S_2, S_3 in R and components D_i of $R - S_i$ containing y_i, for all $i \in \{1, 2, 3\}$, such that D_1, D_2, D_3 are pairwise disjoint and $R - (D_1 \cup D_2 \cup D_3)$ has exactly two components, each containing exactly one vertex from S_i, for all $i \in \{1, 2, 3\}$.

Let $S_i = \{a_i, t_i\}$ for all $i \in \{1, 2, 3\}$ such that $\{a_1, a_2, a_3\}$ is contained in a component A of $R - (D_1 \cup D_2 \cup D_3)$ and $\{t_1, t_2, t_3\}$ is contained in a component T of $R - (D_1 \cup D_2 \cup D_3)$.

Note that each TK_5 we found in Case 2 uses b to connect y_1 and y_2, which can be done in this case by using T. So by treating T, A here as $b, R' - b$ in Case 2, respectively, the arguments in Case 2 work for Case 3 as well and produce a TK_5 in G. ∎

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