Forbidden subgraphs and 3-colorings

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Abstract

A graph G is said to satisfy the Vizing bound if $\chi(G) \leq \omega(G) + 1$, where $\chi(G)$ and $\omega(G)$ denote the chromatic number and clique number of G, respectively. The class of graphs satisfying the Vizing bound is clearly χ-bounded in the sense of Gyárfás. It has been conjectured that if G is triangle-free and fork-free, where the fork is obtained from $K_{1,4}$ by subdividing two edges, then G satisfies the Vizing bound. We show that this is true if, in addition, G is also C_5-free.

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1 Introduction

A class of graphs is said to be χ-*bounded*, with *binding function* f, if for every graph G in this class, $\chi(G) \leq f(\omega(G))$, where $\chi(G)$ and $\omega(G)$ denote the chromatic number and clique number of G, respectively. This terminology was introduced by Gyárfás [5]; see [12] for more contents and references. Note that the class of all graphs that are χ-bounded with binding function $f(x) = x$ contains the famous class of perfect graphs, and perfect graphs have been well characterized, see [2].

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Let $\mathcal{H}$ be a family of graphs. A graph G is said to be *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$ then we use H-free instead of $\mathcal{H}$-free. Gyárfás [4] and Sumner [13] independently conjectured that if F is a forest then the class of F-free graphs is χ-bounded. While this conjecture remains open, Kierstead and Penrice [9] proved it when F is a tree of radius two (extending the techniques of Gyárfás, Szemerédi and Tuza [6] for triangle-free graphs), and more recently Kierstead and Zhu [10] proved it for certain radius three trees.

Here we are interested in the question raised in [12] that for which trees T, T-free graphs are χ-bounded with binding function $f(x) = x + 1$? This is related to Vizing's theorem on chromatic index $\chi'(G)$. Let G be a graph and let $L(G)$ denote the line graph of G. It is easy to see that $\chi'(G) = \chi(L(G))$ and $\Delta(G)) = \omega(L(G))$ (except when $G = K_3$). Thus Vizing's edge coloring theorem implies that $\chi(L(G)) \leq \omega(L(G)) + 1$. Hence, we say that a graph G satisfies the *Vizing bound* if $\chi(G) \leq \omega(G) + 1$. So line graphs satisfy the Vizing bound, or equivalently line graphs form a χ-bounded class with binding function $f(x) = x + 1$.

Beineke [1] showed that a graph is the line graph of some graph iff it does not contain any of a list of nine small graphs as an induced subgraph. The graphs $K_{1,3}$ and $K_5 - e$ are two members of Beineke's list. Kierstead [8] showed that if a graph G contains neither $K_{1,3}$ nor $K_5 - e$ as an induced subgraph then G satisfies the Vizing bound. Thus, using the terminology of Randerath [11], $(K_5 - e, K_{1,3})$ is a Vizing pair. A pair (A, B) of connected graphs is said to be a *good Vizing pair* if every $\{A, B\}$-free graph satisfies the Vizing bound, and neither "A-free" nor "B-free" is redundant. A good Vizing pair (A, B) is *saturated*, if for every good Vizing pair (A', B') with $A \subseteq A'$ and $B \subseteq B'$ we have $A \cong A'$ and $B \cong B'$; see [11].

Randerath [11] studied the first nontrivial case for good Vizing pairs: Determine all pairs (A, B) of connected graphs such that $\{A, B\}$-free graphs are 3-colorable. Obviously, $K_4 \not\subseteq G$. Thus, let B be an induced subgraph of K_4, and there are two nontrivial cases: $B \cong K_4$ and $B \cong K_3$. Now A must be a forest if (A, B) is a good pair, by the result of Erdős and Hajnal [3] that for any positive integers k, ℓ there exist graphs G with $\chi(G) > k$ and girth larger than ℓ. Randerath proved that (K_4, P_4) is a good Vizing pair which is saturated. Thus, it remains to deal with triangle-free graphs which are 3-colorable. Randerath (see [12]) proved that (K_3, H), (K_3, E), and (K_3, cross) are also good Vizing pairs, where H is the connected graph with two vertices of degree 3 and four vertices of degree 1, E is the graph obtained from $K_{1,3}$ by subdividing two edges (each exactly once), and a cross is the graph obtained from $K_{1,4}$ by subdividing one edge (exactly once).

A *fork* is the graph obtained from $K_{1,4}$ by subdividing two edges (each exactly once). The fork with vertex set $\{u, u_1, u_2, v_1, v_2, v_3, v_4\}$ and edge set $\{uu_1, uu_2, uv_1, v_1v_2, uv_3, v_3v_4\}$ is denoted as $(uu_1, uu_2, uv_1v_2, uv_3v_4)$. To obtain a complete characterization of all saturated pairs (K_3, A), it remains to settle the case (K_3, fork) and Randerath [11] (also see [12]) made the following

Conjecture 1.1. *Let G be a triangle-free and fork-free graph. Then $\chi(G) \leq \omega(G) + 1 \leq 3$.*

The main result of this paper is that Conjecture 1.1 holds for graphs with odd girth $og(G) \geq 7$, i.e., the length of a shortest odd cycle is at least 7.

Theorem 1.2. *Let G be a graph such that G is fork-free and $og(G) \geq 7$. Then $\chi(G) \leq 3$.*

We first prove Theorem 1.2 for a special class of graphs G in which there exists a shortest odd cycle C such that all vertices of G are within distance two from C. This is done in Section 2. The remainder of this paper is then devoted to the arguments showing that a minimum counterexample to Theorem 1.2 must belong to this special class; hence, a contradiction. Below we give an outline of these arguments. Let G be a counterexample to Theorem 1.2 with $|V(G)|$ minimum. Then G is fork-free, $og(G) \geq 7$, and $\chi(G) \geq 4$.

In Section 3, we show that certain configurations (or subgraphs) are reducible, i.e., if G contains such a configuration then we can get a smaller fork-free graph H with $og(H) \geq 7$ such that $\chi(H) \leq 3$ would imply $\chi(G) \leq 3$.

In Section 4, we show that G has a shortest odd cycle C with some vertex of degree at least 4 in G, and show that any 4-cycle in G contains more than one vertex of degree at least 4 in G. Both results facilitate the use of "fork-freeness" in Sections 5 and 6, where we show that G has a shortest odd cycle C satisfying certain properties.

In Section 7, we determine the structure of G: G has two subgraphs H and K, such that $G = H \cup K$, K contains a shortest odd cycle C of G, all vertices of K are within distance two of C, $S := H \cap K$ is contained in $V(C)$ and consists of vertices of degree 3 in G. By the minimality of G, H has a 3-coloring which induces a 3-coloring on S. In Section 2, we show that the 3-coloring on S can be extended to K, which means that G is 3-colorable, a contradiction.

For a more detailed overview of the proof of Theorem 1.2 we refer the interested reader to take a glance at Section 7 after reading the notation given in the next paragraph.

We end this section with some notation. Let G be a graph. For any $S \subseteq V(G)$, we use G/S to denote the graph obtained from G by identifying S to a single vertex. Let $x, y \in V(G)$; if x is adjacent to y we write $x \sim y$, and otherwise we write $x \nsim y$. For $x \in V(G)$, let $N_G(x) = \{y \in V(G) : y \sim x\}$ and let $d_G(x) = |N_G(x)|$. For any positive integer k, let $V_k(G) = \{v \in V(G) : d(v) = k\}$. If G is understood, we drop the reference to G. Let $S \subseteq V(G) \cup E(G)$; then $G - S$ denotes the graph obtained from G by deleting S as well as all edges of G incident with $S \cap V(G)$. If $S = \{s\}$ then we simply write $G - s$ for $G - S$. For any $H \subseteq G$, let $G - H = G - V(H)$, let $G[H]$ denote the subgraph of G induced by H, and let $N_i(H)$ denote the set of vertices of G of distance i from H. We use $v_1 \dots v_k v_1$ to represent the cycle C_k with vertex set $\{v_i : 1 \leq i \leq k\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq k-1\} \cup \{v_k v_1\}$.

2 Weakly dominating cycle

Let G be a fork-free graph with $og(G) \geq 7$, and let $C = v_1 \dots v_g v_1$ be a shortest odd cycle in G. Suppose $V(G) = V(C) \cup N_1(C) \cup N_2(C)$ (in this case, C is called a *weakly dominating* cycle in G). Moreover, assume that for any $u \in N_2(C)$ there exist $1 \leq i \leq g$ and two paths uu_1v_{i-1} and uu_1v_{i+1} (and u is said to be *associated with* v_i). All operations in the subscript are modulo g. We derive properties (1)–(6) below about the structure of G.

(1) If $u \in N_2(C)$ is associated with v_i and if $w \in N(u) \cap N_1(C)$, then $N(w) \cap V(C) \subseteq \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$.

Let uu_1v_{i-1}, uu_1v_{i+1} be paths, and suppose there exist $w \in N(u) \cap N_1(C)$ and $v_j \in$

$N(w) \setminus \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$. Then since $og(G) \geq 7$ and C is a shortest cycle in G, $w \neq u_1$ and $w \nsim \{v_{i-2}, v_i, v_{i+2}\}$. Then either $wuu_1v_{i+1}v_{i+2} \dots v_{j-1}v_jw$ or $wuu_1v_{i-1}v_{i-2} \dots v_{j+1}v_jw$ is an odd cycle shorter than C, a contradiction.

(2) If $u \in N_2(C)$ is associated with v_i and v_j and $v_i \neq v_j$, then $v_j \in \{v_{i-2}, v_{i+2}\}$.

For, suppose u is associated with v_i and v_j, and let uu_1v_{i-1}, uu_1v_{i+1}, uw_1v_{j-1}, and uw_1v_{j+1} be paths. By (1), $v_{j-1}, v_{j+1} \in \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$; so $v_j \in \{v_{i-2}, v_{i+2}\}$.

(3) No v_i can be associated with two distinct vertices in $N_2(C)$.

Suppose $u, w \in N_2(C)$ such that $u \neq w$ and u, w are associated with v_i. Let uu_1v_{i-1}, uu_1v_{i+1}, ww_1v_{i-1}, and ww_1v_{i+1} be paths in G. Then $u_1 \nsim w$ to avoid the fork (u_1u, u_1w, $u_1v_{i-1}v_{i-2}$, $u_1v_{i+1}v_{i+2}$). Similarly, $w_1 \nsim u$. Hence, $u_1 \neq w_1$. Now it is easy to see that $(v_{i+1}v_i, v_{i+1}v_{i+2}, v_{i+1}u_1u, v_{i+1}w_1w)$ is a fork in G, a contradiction.

(4) Let $u, w \in N_2(C)$ be associated with v_i, v_j, respectively, and $u \neq w$. Then $v_j \notin \{v_{i-2}, v_i, v_{i+2}\}$, and if $u \sim w$ then $v_j \in \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$.

Let $uu_1v_{i-1}, uu_1v_{i+1}, ww_1v_{j-1}, ww_1v_{j+1}$ be paths in G. By (3), $v_j \neq v_i$, $u_1 \nsim w$ and $w_1 \nsim u$. If $v_j \in \{v_{i-2}, v_{i+2}\}$ then by symmetry let $v_j = v_{i+2}$; now $u_1 \neq w_1$ by the minimality of C, and hence $(v_{i+1}v_i,\ v_{i+1}v_{i+2},\ v_{i+1}u_1u, v_{i+1}w_1w)$ is a fork, a contradiction. So $v_j \notin \{v_{i+2}, v_{i-2}\}$. Suppose $u \sim w$ and $v_j \notin \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$, and without loss of generality, assume $1 \leq i < j \leq g$. Then either $uu_1v_{i-1}v_{i-2} \dots v_{j+2}v_{j+1}w_1wu$ (when $j - i$ is odd) or $uu_1v_{i+1}v_{i+2} \dots v_{j-2}v_{j-1}w_1wu$ (when $j - i$ is even) is an odd cycle shorter than C, a contradiction.

(5) Each component of $G[N_2(C)]$ is a path, and if $x_1x_2x_3 \dots x_t$, $t \geq 3$, is a component of $G[N_2(C)]$, then $t \leq 4$ and (by relabeling $x_1x_2 \dots x_t$ if necessary) we may assume that for some $1 \leq i \leq g$, x_1, x_2 are associated with v_i, v_{i+1}, respectively, x_3 is associated with v_{i+4}, and if $t = 4$ then x_4 is associated with v_{i+5}.

First, let $x_1x_2x_3$ be an arbitrary path in $G[N_2(C)]$, and assume that x_1, x_2, x_3 are associated with v_i, v_j, v_k, respectively. By (4) and by symmetry, we may assume $v_j \in \{v_{i+1}, v_{i+3}\}$. Then by (4) and (3), $v_k \in \{v_{j+1}, v_{j+3}\}$, and if $v_j = v_{i+1}$ then $v_k = v_{j+3}$. If $v_j = v_{i+3}$ then $v_k = v_{j+1}$; for if $v_k = v_{j+3}$ then letting $x_1u_1v_{i-1}, x_3w_1v_{k+1}$ be two paths in G, we see that $x_1u_1v_{i-1}v_{i-2} \dots v_{k+2}v_{k+1}w_1x_3x_2x_1$ is an odd cycle shorter than C, a contradiction. So by symmetry, we may assume $v_j = v_{i+1}$ and $v_k = v_{j+3} = v_{i+4}$. By (3) and (4), x_2 has degree 2 in $G[N_2(C)]$.

Therefore, $\Delta(G[N_2(C)]) \leq 2$. Hence each component of $G[N_2(C)]$ is a path or a cycle. Thus, if all components of $G[N_2(C)]$ have at most three vertices, (5) holds. So let $D = x_1x_2x_3 \dots x_t$ be a path in $G[N_2(C)]$, with $t \geq 4$.

Suppose there exists i such that x_2, x_3 are associated with v_i, v_{i+1}, respectively. Then by applying the above conclusion on $x_1x_2x_3$ to $x_1x_2x_3$ and $x_2x_3x_4$, x_1, x_4 are associated with v_{i-3}, v_{i+4}, respectively. Let x_1uv_{i-4} and x_4wv_{i+5} be two paths in G. Then $x_1x_2x_3x_4wv_{i+5}v_{i+6} \dots v_{i-5}v_{i-4}ux_1$ is an odd cycle shorter than C, a contradiction.

Thus, by the above conclusion for $x_1x_2x_3$, we may assume that x_1, x_2 are associated with v_1, v_2, respectively. Then by (3) and (4) and by applying the above conclusion for $x_1x_2x_3$ to $x_2x_3x_4$, x_3, x_4 are associated with v_5, v_6, respectively. If $t = 4$ and $x_1x_4 \in E(G)$ then let

x_1uv_g and x_4wv_7 be two paths in G; now $x_1x_4wv_7v_8\dots v_gux_1$ is an odd cycle shorter than C, a contradiction. If $t \geq 5$ then by applying the above conclusion for $x_1x_2x_3$ to $x_3x_4x_5$, x_5 is associated with v_9. Let x_1uv_g and x_5wv_{10} be two paths in G. Now $x_1x_2x_3x_4x_5wv_{10}v_{11}\dots v_gux_1$ is an odd cycle shorter than C, a contradiction. So $t = 4$ and $x_1 \nsim x_4$. Hence (5) holds.

Our objective is to produce a 3-coloring of G, with certain vertices of C precolored. For this, we divide the neighbors of each v_i not on C into several groups. Let

$$
\begin{aligned}
X_{i,1} &:= \{v \in N(v_i) \setminus V(C) : N(v) \cap V(C) = \{v_i\} \text{ and } N(v) \cap N(\{v_{i-3}, v_{i+3}\}) = \emptyset\},\\
X_{i,2}^{+} &:= (N(v_i) \cap N(v_{i+2})) \setminus V(C),\\
X_{i,2}^{-} &:= (N(v_i) \cap N(v_{i-2})) \setminus V(C),\\
X_{i,2} &:= X_{i,2}^{+} \cup X_{i,2}^{-},\\
X_{i,3}^{+} &:= \{v \in N(v_i) \setminus V(C) : N(v) \cap (N(v_{i+3}) \setminus V(C)) \neq \emptyset\},\\
X_{i,3}^{-} &:= \{v \in N(v_i) \setminus V(C) : N(v) \cap (N(v_{i-3}) \setminus V(C)) \neq \emptyset\},\\
X_{i,3} &:= X_{i,3}^{+} \cup X_{i,3}^{-}
\end{aligned}
$$

Let $X_j := \bigcup_{i=1}^{g} X_{i,j}$, $j = 1, 2, 3$. By definition, $X_1 \cap (X_2 \cup X_3) = \emptyset$.

(6) For $1 \leq i \leq g$, $N(v_i) \setminus V(C) = X_{i,1} \cup X_{i,2} \cup X_{i,3}$, $|X_{i,1}| \leq 1$, and $X_{i,j}^{+} \cap X_{i,k}^{-} = \emptyset$ for $j, k \in \{2, 3\}$.

Let $v \in N(v_i) \setminus V(C)$ such that $v \notin X_{i,1}$. If $N(v) \cap N(\{v_{i-3}, v_{i+3}\}) \neq \emptyset$ then by definition, $v \in X_{i,3}$. So assume $N(v) \cap N(\{v_{i-3}, v_{i+3}\}) = \emptyset$. Then $N(v) \cap V(C) \neq \{v_i\}$ as $v \notin X_{i,1}$. By the minimality of C, $N(v) \cap \{v_{i-2}, v_{i+2}\} \neq \emptyset$; so $v \in X_{i,2}$. Thus, $N(v_i) = X_{i,1} \cup X_{i,2} \cup X_{i,3}$.

If $|X_{i,1}| \geq 2$ and $x, y \in X_{i,1}$ are distinct, then $(v_ix, v_iy, v_iv_{i-1}v_{i-2}, v_iv_{i+1}v_{i+2})$ is a fork, a contradiction; so $|X_{i,1}| \leq 1$. Finally, it is easy to check, using the minimality of C, that $X_{i,j}^{+} \cap X_{i,k}^{-} = \emptyset$ for $1 \leq i \leq g$ and $j, k \in \{2, 3\}$. This proves (6).

Lemma 2.1. *Let G be a fork-free graph with $og(G) \geq 7$, and let $C = v_1 \dots v_gv_1$ be a shortest odd cycle in G such that $V(G) = V(C) \cup N_1(C) \cup N_2(C)$ and each vertex in $N_2(C)$ is associated with some v_i. Let $S := V_2(G) \cap V(C)$ such that*

- *(i) if some vertex in $N(v_i)$ is adjacent to two vertices in $N_2(C)$, one associated with one of $\{v_{i-3}, v_{i-1}\}$ and the other associated with one of $\{v_{i+3}, v_{i+1}\}$, then $v_{i-1}, v_{i+1} \notin S$,*
- *(ii) if $X_{i,1} \neq \emptyset$ and $v_j \in \{v_{i-1}, v_{i+1}\} \cap S$ then v_j is not associated with any vertex in $N_2(C)$,*
- *(iii) if v_i is associated with some vertex in $N_2(C)$ which is adjacent to some vertex in $X_{i+1,1} \cup X_{i+1,2}^{+} \cup X_{i+1,3}^{+}$ (respectively, $X_{i-1,1} \cup X_{i-1,2}^{-} \cup X_{i-1,3}^{-}$) then $v_i \notin S$ or $v_{i+3} \notin S$ (respectively, $v_{i-3} \notin S$).*

Then any 3-coloring of $G[S]$ can be extended to a 3-coloring of G.

Proof. Let $c_S : S \to \{1, 2, 3\}$ be a 3-coloring of $G[S]$. We now extend c_S to a 3-coloring c of G in four steps: color C first, then $X_2 \cup X_3$, then X_1, and finally $N_2(C)$.

Step 1. If $S \neq \emptyset$ we simply extend c_S to a 3-coloring c of C so that each component of $C - S$ is 2-colored. If $S = \emptyset$ then, since $|C|$ is odd and by (3), (4) and (5), there exists some v_i such that v_{i-1}, v_{i+1} are not associated with any vertex in $N_2(C)$. Without loss of generality, assume that v_2, v_g are not associated with any vertex in $N_2(C)$. Let c be the 3-coloring of C such that $c(v_1) = 3$ and, for $2 \leq i \leq g$, $c(v_i) = 1$ if i is even, and $c(v_i) = 2$ if i is odd.

Step 2. We extend c to a 3-coloring of $G[V(C) \cup X_2 \cup X_3]$ as follows: for each $v \in X_{i,2}^+ \cup X_{i,3}^+$, let $c(v) = c(v_{i+1})$; and for each $v \in X_{i,2}^- \cup X_{i,3}^-$, let $c(v) = c(v_{i-1})$.

By (6), we have $X_{i,j}^+ \cap X_{i,k}^- = \emptyset$ for $j, k \in \{2, 3\}$; so c is well defined. We now prove that c is a 3-coloring of $G[V(C) \cup X_2 \cup X_3]$.

First, we show that for any $v \in X_2 \cup X_3$, if $v \sim v_i$ for some $1 \leq i \leq g$ then $c(v) \neq c(v_i)$. Assume $v \in X_2$. Then $v \in X_{j,2}^+$ for some $1 \leq j \leq g$. So by the minimality of C, $N(v) \cap V(C) = \{v_j, v_{j+2}\}$ and $i \in \{j, j+2\}$. Hence $c(v) = c(v_{j+1}) \neq c(v_i)$. Now assume $v \notin X_2$. By symmetry we may assume $v \in X_{j,3}^+$ and $i = j$. Then $c(v) = c(v_{j+1}) \neq c(v_i)$.

Next, we show that $c(v) \neq c(w)$ for any $v, w \in X_2 \cup X_3$ with $vw \in E(G)$. First, assume $v, w \in X_2$, and let $v \sim v_i$, $v \sim v_{i+2}$, $w \sim v_j$ and $w \sim v_{j+2}$ such that $1 \leq i \leq j < j+2 \leq g$. Since $og(G) \geq 7$, $\{v_i, v_{i+2}\} \cap \{v_j, v_{j+2}\} = \emptyset$. If $v_j = v_{i+1}$ then $c(v) = c(v_{i+1}) \neq c(v_{i+2}) = c(w)$. Hence, assume $v_j \notin \{v_i, v_{i+1}, v_{i+2}\}$. Then $vv_{i+2}v_{i+3} \dots v_{j-1}v_jwv$ (when $j - i$ is even) or $vv_iv_{i-1} \dots v_{j+3}v_{j+2}wv$ (when $j - i$ is odd) is an odd cycle shorter than C, a contradiction.

Thus by symmetry, let $w \in X_{j,3}^+$ with $w' \in N(w) \cap N(v_{j+3})$. If $v \sim v_{j+3}$ then $v \in X_{j+3,3}^-$ (since $v \sim w$); so $c(v) = c(v_{j+2}) \neq c(v_{j+1}) = c(w)$; and if $v \sim v_{j+1}$ then $c(v) \neq c(v_{j+1}) = c(w)$. So assume $v \nsim \{v_{j+1}, v_{j+3}\}$. Note that $v \nsim \{v_j, v_{j+2}\}$ as $og(G) \geq 7$. Let $v \sim v_i$; so $v_i \notin \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$. Since $v \in X_2 \cup X_3$, we may further choose v_i so that $v_i \notin \{v_{j-1}, v_{j-2}\}$. By symmetry, let $1 \leq j+3 < i \leq g$. Then either $vwv_jv_{j-1} \dots v_{i+1}v_iv$ (when $i - (j+3)$ is even) or $vww'v_{j+3}v_{j+4} \dots v_{i-1}v_iv$ (when $i - (j+3)$ is odd) is an odd cycle shorter than C, a contradiction.

Step 3. We further extend c to a 3-coloring of $G[V(C) \cup N_1(C)]$ by coloring vertices in X_1. A *band* in G is a maximal sequence $v_sw_sv_{s+1}w_{s+1} \dots v_tw_t$ such that $w_i \in X_{i,1}$ for $i = s, s+1, \dots, t$, and $w_i \sim w_{i+1}$ for $i = s, s+1, \dots, t-1$. Let $v_sw_sv_{s+1}w_{s+1} \dots v_tw_t$ be a band.

Suppose $S \neq \emptyset$. Let $c(w_i) = c(v_{i-1})$ (respectively, $c(v_{i+1})$) when w_i is adjacent to some vertex in $N_2(C)$ that is associated with some $v_j \in \{v_{i-3}, v_{i-1}\}$ (respectively, $v_j \in \{v_{i+1}, v_{i+3}\}$); and otherwise let $c(w_i) = c(v_{i-1})$ for $i = s+1, \dots, t$, $c(w_s) = c(v_{s+1})$ when $t \neq s$, and if $t = s$ then $c(w_t) = c(v)$ for some $v \in \{v_{t-1}, v_{t+1}\}$ with $v \notin S$ whenever possible. Now c is well defined, as by (i) and by the coloring in Step 1, $c(v_{i-1}) = c(v_{i+1})$ if one of $\{v_{i-1}, v_{i-3}\}$ and one of $\{v_{i+1}, v_{i+3}\}$ are associated with vertices in $N_2(C)$.

If $S = \emptyset$ then let $c(w_i) = 3$ if $i \in \{2, g\}$, $c(w_i) = 1$ if $i = 1$ and $N(w_i) \cap (X_{i+1,2}^+ \cup X_{i+1,3}^+) \neq \emptyset$, $c(w_i) = 2$ if $i = 1$ and $N(w_i) \cap (X_{i-1,2}^- \cup X_{i-1,3}^-) \neq \emptyset$, and $c(w_i) = c(v_{i-1})$ for all other i. By the minimality of C, $N(w_i) \cap (X_{i+1,2}^+ \cup X_{i+1,3}^+) = \emptyset$ or $N(w_i) \cap (X_{i-1,2}^- \cup X_{i-1,3}^-) = \emptyset$, so c is well defined. Note that with the possible exceptions of w_g, w_1, w_2, all colors $c(w_i)$ alternate between 1 and 2.

We now show that c is a proper coloring of $G - N_2(C)$. By (6), if $v_i \sim w_i \in X_1$ then $X_{i,1} = \{w_i\}$ and $c(v_i) \neq c(w_i)$ as $c(w_i) \in \{c(v_{i-1}), c(v_{i+1})\}$ by definition.

Note that if two vertices in X_1 are adjacent, they are in a band, say $v_sw_sv_{s+1}w_{s+1} \dots v_tw_t$.

We prove $c(w_i) \neq c(w_{i+1})$. First, assume $S = \emptyset$. If $v_i = v_1$ then $c(w_{i+1}) = 3 = c(v_i) \neq c(w_i)$; and if $v_{i+1} = v_1$ then $c(w_i) = 3 = c(v_{i+1}) \neq c(w_{i+1})$. So by symmetry assume $v_1 \notin \{v_i, v_{i+1}, v_{i+2}\}$. Then $c(v_i) = c(v_{i+2})$, and hence $c(w_{i+1}) = c(v_i) \neq c(w_i)$. Now assume $S \neq \emptyset$. If $c(w_{i+1}) = c(v_i)$ or $c(v_i) = c(v_{i+2})$ then $c(w_{i+1}) \neq c(w_i)$. So assume $c(w_{i+1}) = c(v_{i+2}) \neq c(v_i)$. Thus by the definition of c, w_{i+1} is adjacent to some $x \in N_2(C)$ associated with v_{i+2} or v_{i+4}. If x is associated with v_{i+2} then by (ii), $v_{i+2} \notin S$; so $c(v_{i+2}) = c(v_i)$, a contradiction. Thus, x is associated with v_{i+4}, and let xuv_{i+5} be a path in G. Similarly, assume $w_i \sim y \in N_2(C)$ which is associated with v_{i-3}, and let yvv_{i-4} be a path in G. By (3), $v_{i-3} \neq v_{i+4}$. Thus $xw_{i+1}w_iyvv_{i-4}v_{i-5}\dots v_{i+6}v_{i+5}ux$ is an odd cycle shorter than C, a contradiction.

Now let $w \in X_2 \cup X_3$, $w \sim w_i$, and w_i contained in some band $v_sw_sv_{s+1}w_{s+1}\dots v_tw_t$. Then $w \nsim \{v_{i-3}, v_{i+3}\}$, as otherwise $w_i \in X_3$, a contradiction. Thus, $w \sim \{v_{i-1}, v_{i+1}\}$ by the minimality of C and by the fact that $og(G) \geq 7$. By symmetry, let $w \sim v_{i+1}$. If $c(w) = c(v_i)$ then $c(w) \neq c(w_i)$. So by the definition of c, assume $c(w) = c(v_{i+2}) \neq c(v_i)$. Thus $w \in X^+_{i+1,3}$ (since $w \nsim v_{i+3}$), and either $v_{i+2} \in S$ or $S = \emptyset$ and $v_1 \in \{v_i, v_{i+1}, v_{i+2}\}$.

Suppose $S \neq \emptyset$. Then $c(w_i) \in \{c(v_{i-1}), c(v_{i+1})\}$. If $c(w_i) = c(v_{i+1})$ or $c(v_{i-1}) = c(v_{i+1})$ then $c(w_i) \neq c(w)$. So assume $c(w_i) = c(v_{i-1}) \neq c(v_{i+1})$. Thus, $v_{i-1} \in S$. By (ii), v_{i-1} is not associated with any vertex in $N_2(C)$. By the minimality of C, v_{i-3} is not associated with any vertex in $N_2(C)$ that is adjacent to w. So by the coloring in Steps 1 and 3, $c(w_i) = c(v_{i+1})$, a contradiction.

Now assume $S = \emptyset$ and $v_1 \in \{v_i, v_{i+1}, v_{i+2}\}$. By definition, if $v_i = v_1$ then $c(w_i) = 1$ and $c(w) = c(v_{i+2}) = 2$, if $v_{i+1} = v_1$ then $c(w_i) = 3$ and $c(w) = c(v_{i+2}) = 1$, and if $v_{i+2} = v_1$ then $c(w) = 3$ and $c(w_i) = 2$. Hence, $c(w_i) \neq c(w)$.

Step 4. We now extend c to a 3-coloring of G by coloring vertices in $N_2(C)$. Let $u \in N_2(C)$ be associated with v_i. By the minimality of C, $N(u) \cap (N(v_{i+3}) \cup X^+_{i+1,3}) = \emptyset$ or $N(u) \cap (N(v_{i-3}) \cup X^-_{i-1,3}) = \emptyset$, if $w \in N(u) \cap N(v_{i+3})$ then $w \in X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$, and if $w \in N(u) \cap N(v_{i-3})$ then $w \in X_{i-3,1} \cup X^+_{i-3,2} \cup X^+_{i-3,3}$.

Case 1. $S = \emptyset$.

Let $u \in N_2(C)$ be associated with v_i, and $w \in N(u) \cap N_1(C)$. By (1), $N(w) \cap V(C) \subseteq \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$. By the choice of v_1 in Step 1, $v_1 \notin \{v_{i-1}, v_{i+1}\}$.

Suppose $v_i = v_1$. We prove $|c(N(u) \cap N_1(C))| \leq 2$. By definition of c, if $w \sim v_{i+3}$ then $c(w) \in \{c(v_{i+2}), c(v_{i+4})\} = \{2\}$; if $w \sim v_{i+1}$ then $c(w) = 3$ (when $w \in X_1$) and $c(w) \in \{c(v_i), c(v_{i+2})\} = \{2, 3\}$ (when $w \in X_2 \cup X_3$), if $w \sim v_{i-3}$ then $c(w) = 1$, and if $w \sim v_{i-1}$ then $c(w) = 3$ (when $w \in X_1$) and $c(w) \in \{1, 3\}$ (when $w \in X_2 \cup X_3$). Therefore, if $w \in N(v_{i+3}) \cup X^+_{i+1,3}$ then $N(u) \cap (N(v_{i-3}) \cup X^-_{i-1,3}) = \emptyset$, and hence, $|c(N(u) \cap N_1(C))| \leq 2$. Similarly, if $w \in N(v_{i-3}) \cup X^-_{i-1,3}$ then $|c(N(u) \cap N_1(C))| \leq 2$. So assume $N(u) \cap (N(v_{i+3}) \cup X^+_{i+1,3} \cup N(v_{i-3}) \cup X^-_{i-1,3}) = \emptyset$. Then $c(w) = 3$ for all $w \in N(u) \cap N_1(C)$; so $|c(N(u) \cap N_1(C))| = 1$.

Now assume that $v_i \neq v_1$ and, by symmetry, assume that $1 < i < g - i$. Then by definition of c, if $w \sim \{v_{i+1}, v_{i+3}\}$ then $c(w) = c(v_{i+2}) = c(v_i)$. Similarly, if $v_{i-2} \neq v_1$, then $c(w) = c(v_i)$ when $w \sim \{v_{i-1}, v_{i-3}\}$. So when $v_{i-2} \neq v_1$, we have $|c(N(u) \cap N_1(C))| = 1$. Now assume $v_{i-2} = v_1$. Then $c(w) = 3$ when $w \in X_{i-1,1} \cup X_{i-3,1} \cup X^-_{i-1,2} \cup X^-_{i-1,3} \cup X^+_{i-3,2} \cup X^+_{i-3,3}$. So $|c(N(u) \cap N_1(C))| = 1$ or $c(N(u) \cap N_1(C)) \subseteq \{2, 3\}$.

Suppose $|c(N(u) \cap N_1(C))| = 2$ and there is $x \in N_2(C) \backslash \{u\}$ such that $|c(N(x) \cap N_1(C))| = 2$. We show that u, x cannot be contained in the same component of $G[N_2(C)]$. By (4), x cannot be associated with $\{v_{i-2}, v_i, v_{i+2}\}$; so by the argument in the previous paragraph and by symmetry we may assume that $v_{i-2} = v_1$, and x is associated with v_{i-4}. Then v_{i-1}, v_{i-3} are not associated with any of $N_2(C)$. Suppose u and x are are contained in some component $x_1 \ldots x_t$ of $G[N_2(C)]$. Then by (5), $u \nsim x$, $x_1 = u$ and $x_t = x$. Since $|V(C)|$ is odd, $t = 4$. Thus, $G[N_2(C)] = x_1x_2x_3x_4$. Hence, $c(N(u) \cap N_1(C)) = \{2, 3\}$, $c(N(x) \cap N_1(C)) = \{1, 3\}$, $c(N(x_2) \cap N_1(C)) = \{1\}$, and $c(N(x_3) \cap N_1(C)) = \{2\}$. Clearly, the coloring c can be extended to G by letting $c(x_1) = c(x_3) = 1$ and $c(x_2) = c(x_4) = 2$.

Let $P_1, \ldots, P_k$ be the components of $G[N_2(C)]$. Then each P_i contains at most one vertex u_i such that $|c(N(u_i) \cap N_1(C))| > 1$ (if no such vertex exists, let $u_i \in V(P_i)$ be arbitrary). Let Q_i and R_i be the subpaths of P_i from u_i to the two ends of P_i, respectively. Now c can be extended to a 3-coloring of G by, for each i, coloring u_i first and then coloring Q_i and R_i greedily in the order from u_i towards their ends.

Case 2. $S \neq \emptyset$.

Let $x_1x_2 \ldots x_t$ be a component of $G[N_2(C)]$, and assume that x_1 is associated with v_i. Let $x_1u_1v_{i-1}, x_1u_1v_{i+1}$ be two paths in G.

Suppose $1 < j < t$, and let x_j be associated with v_k. Then by (5), $v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2} \notin S$; hence by the coloring of C given in Step 1, $c(v_{k-2}) = c(v_k) = c(v_{k+2})$. By (1), for any $w \in N(x_j) \cap N_1(C)$, $N(w) \cap V(C) \subseteq \{v_{k-3}, v_{k-1}, v_{k+1}, v_{k+3}\}$. By symmetry assume $w \sim \{v_{k+1}, v_{k+3}\}$. If $w \sim v_{k+1}$ then $c(w) \in \{c(v_k), v_{k+2}\}$; so $c(w) = c(v_k)$. If $w \sim v_{k+3}$ then by the minimality of C, $w \in X_{k+3,1} \cup X^-_{k+3,2} \cup X^-_{k+3,3}$; so $c(w) = c(v_{k+2}) = c(v_k)$ by the coloring in Steps 2 and 3. Hence, $|c(N(x_j) \cap N_1(C))| = 1$.

We now investigate $c(N(x_1) \cap N_1(C))$. Let $w \in N(x_1) \cap N_1(C)$. Then as above, by (1) and by the minimality of C and symmetry, we may assume that $w \in X_{i+1,1} \cup X_{i+1,2} \cup X_{i+1,3} \cup X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$.

So by the definition of coloring in Steps 2 and 3, if $w \in X^+_{i+1,2} \cup X^+_{i+1,3}$ then $c(w) = c(v_{i+2})$; if $w \in X_{i+1,1} \cup X^-_{i+1,2} \cup X^-_{i+1,3}$ then $c(w) = c(v_i)$; if $w \in X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$ then $c(w) = c(v_{i+2})$. Thus, since $N(u) \cap (N(v_{i+3}) \cup X^+_{i+1,3}) = \emptyset$ or $N(u) \cap (N(v_{i-3}) \cup X^-_{i-1,3}) = \emptyset$ (by minimality of C), $c(N(x_1) \cap N_1(C)) \subseteq \{c(v_i), c(v_{i+2})\}$.

Therefore, if $t = 1$ then c can be extended by assigning x_1 a color not in $\{c(v_i), c(v_{i+2})\}$; if $c(v_i) = c(v_{i+2})$ then by (5), we can extend c by greedily coloring $x_t, x_{t-1}, \ldots, x_1$ in order. So we may assume $w \notin X_{i+1,1}$, $c(v_i) \neq c(v_{i+2})$ and $t \geq 2$. Then $v_i \in S$ or $v_{i+2} \in S$. Hence, x_2 cannot be associated with v_{i+1} and, by the minimality of C, x_2 is not associated with v_{i-3}.

Suppose x_2 is associated with v_{i-1}. Since $w \in X^+_{i+1,2} \cup X^+_{i+1,3} \cup X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$, it follows from (5) and the minimality of C that $t = 2$ and $N(x_2) \cap N(v_{i-4}) = \emptyset$; so $c(N(x_2) \cap N_1(C)) = \{c(v_{i-1})\}$ (as $c(v_{i-1}) = c(v_{i+1})$). Thus, c can extended to x_1x_2 by greedily coloring x_1 and then x_2.

So assume that x_2 is associated with v_{i+3}. Then $v_{i+2} \notin S$ and $v_i \in S$. By (5), $t = 2$ or $t = 3$. If $t = 2$ then $v_{i+3} \notin S$ (otherwise $w \in X^+_{i+1,3}$, contradicting (iii)); and if $t = 3$ then by (5), x_3 is associated with v_{i+4}. By applying the above argument for x_1 to x_2, we may assume that $|c(N(x_2) \cap N_1(C)| = 1$. So c can be extended to $x_1 \ldots x_t$ by greedily coloring $x_1, \ldots, x_t$

in order.

Therefore, c can be extended to color each component of $G[N_2(C)]$, and hence to a 3-coloring of G. ∎

By setting $S = V(C) \cap V_2(G)$ and $N_2(C) = \emptyset$ in Lemma 2.1, we have the following

Corollary 2.2. *Let G be a fork-free graph with $og(G) \geq 7$, let $C = v_1 \dots v_g v_1$ be a shortest odd cycle in G such that $V(G) = V(C) \cup N_1(C)$, and let $S = V(C) \cap V_2(G)$. Then any 3-coloring of $G[S]$ can be extended to a 3-coloring of G.*

3 Properties of a minimum counterexample

First, we state a generalization of Brooks' Theorem due to Gallai [7].

Theorem 3.1 (Gallai)**.** *Let G be a k-vertex-critical graph and $Low(G)$ denote the subgraph of G induced by the vertices of degree $k-1$ in G. Then every 2-connected induced subgraph of $Low(G)$ is either a complete graph or an odd cycle of length at least 5.*

Suppose the assertion of Theorem 1.2 fails. Then we may choose a graph G such that

(1) G is fork-free and $og(G) \geq 7$,

(2) $\chi(G) \geq 4$, and

(3) subject (1) and (2), $|V(G)|$ is minimum.

Lemma 3.2. *G is 4-color critical, every 2-connected induced subgraph of $G[V_3]$ is either an odd cycle of length at least 7 or a complete graph, and $N(u) \not\subseteq N(v)$ for any distinct $u, v \in V(G)$.*

Proof. By the minimality of G, $\chi(G - v) \leq 3$ for any $v \in V(G)$. But $\chi(G) \geq 4$, so G is 4-color critical. Thus, the second part of the assertion follows from Theorem 3.1. For the third part, let $u, v \in V(G)$ be distinct such that $N(u) \subseteq N(v)$. Let c be a 3-coloring of $G - v$. Assigning $c(v)$ to the vertex u, we extend c to a 3-coloring of G, a contradiction. ∎

We may view $u, v \in V(G)$ with $N(u) \subseteq N(v)$ as forming a *reducible configuration*, which does not exist in G by Lemma 3.2. The next two results exclude from G two more reducible configurations.

Lemma 3.3. *Let $u_1, v_1, u_2, v_2 \in V(G)$ be distinct such that $u_i \sim v_i$ and $u_i \nsim v_{3-i}$ for $i = 1, 2$, and $u_1 \nsim u_2$ and $v_1 \nsim v_2$. If $N(u_1) \setminus \{v_1\} \subseteq N(u_2) \setminus \{v_2\}$ then $N(v_1) \setminus \{u_1\} \not\subseteq N(v_2) \setminus \{u_2\}$.*

Proof. For, suppose $N(u_1) \setminus \{v_1\} \subseteq N(u_2) \setminus \{v_2\}$ and $N(v_1) \setminus \{u_1\} \subseteq N(v_2) \setminus \{u_2\}$. By the minimality of G, $G - \{u_1, v_1\}$ admits a 3-coloring, say c. Now c can be extended to a 3-coloring of G by assigning $c(u_2)$ to u_1 and $c(v_2)$ to v_1. ∎

Lemma 3.4. *Let $v, w \in V(G)$ and $N(w) = \{v, w_1, \cdots, w_k\}$, with $k \geq 3$, such that $|N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})| \leq 1$. Then there exists $x \in N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})$ such that $|N(x) \cap \{w_1, \cdots, w_k\}| = 1$, or there exists $x \in N(\{w_1, \cdots, w_k\}) \setminus N(v)$ such that $|N(x) \cap \{w_1, \cdots, w_k\}| \leq k - 2$.*

Proof. Suppose for all $x \in N(\{w_1, \cdots, w_k\}) \setminus N(v)$, $|N(x) \cap \{w_1, \cdots, w_k\}| \geq k-1$, and if there exists $x \in N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})$ then $|N(x) \cap \{w_1, \cdots, w_k\}| \geq 2$. Let $N(\{w_1, \cdots, w_k\}) \setminus \{w\} = \{x_1, \cdots, x_s\}$, $G' = (G - w)/\{w_1, \cdots, w_k\}$, and let x denote the identification of $w_1, \cdots, w_k$.

We claim that $og(G') \geq 7$. For suppose T' is a cycle in G' with $|V(T')| = 3$ or 5. Then $x \in V(T')$ as $og(G) \geq 7$. Without loss generality, assume $xx_1, xx_2 \in E(T')$. By the assumption above, there exists some $i \in \{1, 2\}$, such that $|N(x_i) \cap \{w_1, \cdots, w_k\}| \geq k-1$ and $|N(x_{3-i}) \cap \{w_1, \cdots, w_k\}| \geq 2$. Hence, there exists some w_j such that $w_j \sim x_1$ and $w_j \sim x_2$. Now $T := (T' - x) + \{w_j, w_jx_1, w_jx_2\}$ is a cycle in G with $|V(T)| = |V(T')|$, a contradiction.

If G' is also fork-free then by the choice of G, G' has a 3-coloring which induces a 3-coloring c of $G - \{w, w_1, \ldots, w_k\}$. Setting $c(w_i) = c(x)$ for $1 \leq i \leq k$ and letting $c(w)$ be a color not in $\{c(v), c(x)\}$, c is extended to a 3-coloring of G, a contradiction.

Thus, let F' be a fork in G'. Then $x \in V(F')$ as G is fork-free. If $d_{F'}(x) = 1$ then without loss of generality let $xx_1 \in E(F')$ and $w_1 \sim x_1$; now $F := (F' - x) + \{w_1, w_1x_1\}$ is a fork in G, a contradiction. If $d_{F'}(x) = 2$ then let $xx_1, xx_2 \in F'$, and as in the previous paragraph, there exists some j such that $w_j \sim x_1$ and $w_j \sim x_2$; but then $F := (F' - x) + \{w_j, w_jx_1, w_jx_2\}$ is a fork in G, a contradiction. So $d_{F'}(x) = 4$ and, without loss of generality, let $F' = (xx_3, xx_4, xx_1y_1, xx_2y_2)$.

By symmetry between x_1 and x_2, assume $x_2 \nsim v$. By the above assumption there exists $w_i \in \{w_1, w_2\}$, say w_1, such that $w_1 \sim x_1$ and $w_1 \sim x_2$. If $x_3 \sim w_1$ then $F := (F' - x) + \{w_1, w, w_1x_1, w_1x_2, w_1x_3, w_1w\}$ would be a fork in G. So $x_3 \nsim w_1$. Similarly, $x_4 \nsim w_1$. So by the above assumption and without loss of generality, assume $w_2 \sim x_i$ for $i = 2, 3, 4$. If $v \nsim \{x_3, x_4\}$ then $F := (F' - \{x, x_1, y_1\}) + \{v, w_2, w, w_2x_2, w_2x_3, w_2x_4, w_2w, wv\}$ would be a fork in G. Thus, assume $x_4 \sim v$. If there exists some w_j such that $w_j \sim x_i$ for $i = 1, 2, 3$ then $F := (F' - x) + \{w_j, w, w_jx_1, w_jx_2, w_jx_3, w_jw\}$ would be a fork in G. So such w_j does not exist. Then by the above assumption again, we have $w \in V_4$, $x_1 \nsim w_2$, $x_3 \sim w_2$, $x_3 \sim w_3$, $x_2 \sim w_1$, $x_1 \sim w_3$, and $x_2 \nsim w_3$.

Let $y_3 \in N(x_3) \setminus N(x_4)$ and $y_4 \in N(x_4) \setminus N(w)$, which exist by Lemma 3.2. Then $y_3 \sim x_2$ to avoid $(w_2x_4, w_2w, w_2x_2y_2, w_2x_3y_3)$, $y_3 \sim x_1$ to avoid $(x_2y_3, x_2y_2, x_2w_2x_4, x_2w_1x_1)$, and $y_4 \sim \{x_1, x_2\}$ to avoid $(x_4y_4, x_4v, x_4w_2x_2, x_4w_3x_1)$. Moreover, $y_4 \nsim x_1$ to avoid $(x_1y_1, x_1y_3, x_1y_4x_4, x_1w_1w)$. So $y_4 \sim x_2$, and $(x_2y_2, x_2y_3, x_2w_1w, x_2y_4x_4)$ is a fork in G, a contradiction. ∎

We need the following three lemmas concerning cycles in $G[V_3]$. The first is a direct consequence of Lemma 3.2.

Lemma 3.5. *$G[V_3]$ contains no induced even cycles.*

Lemma 3.6. *Let C be an induced cycle in $G[V_3]$. Then for any 3-coloring c of $G - C$ and for any $x, y \in N(C)$, $c(x) = c(y)$.*

Proof. Let c be a 3-coloring of $G - C$, $C = v_1 \cdots v_g v_1$, and $\{w_i\} = N(v_i) \setminus V(C)$ for $i = 1, \cdots, g$. Suppose $c(w_i) \neq c(w_j)$ for some $1 \leq i \neq j \leq g$. Then there exists $s \in \{1, \cdots, g\}$ such that $c(w_s) \neq c(w_{s+1})$. Without loss of generality, let $c(w_1) \neq c(w_2)$. Coloring v_2 by $c(w_1)$ and coloring $v_3, v_4, \cdots, v_g, v_1$ greedily in order, we extend c to a 3-coloring of G, a contradiction. ∎

Lemma 3.7. *Let $C = v_1v_2 \cdots v_g v_1$ be a shortest odd cycle in G, and assume $C \subseteq G[V_3]$. Then*

(1) $(N(v_i) \setminus V(C)) \cap (N(v_j) \setminus V(C)) = \emptyset$*, for* $1 \leq i \neq j \leq g$*;*

(2) $\bigcup_{i=1}^{g}(N(v_i) \setminus V(C))$ *is independent;*

(3) *for* $1 \leq i \leq g$*,* $G - (C - \{v_i, v_{i+1}\})$ *has a path from* v_i *to* v_{i+1} *of length 6.*

Proof. Suppose (1) fails. Then, since $|V(C)| = og(G) \geq 7$, we may assume without loss of generality that $v \in (N(v_1) \setminus V(C)) \cap (N(v_3) \setminus V(C))$. By Lemma 3.5, $v \notin V_3$. So let $v', v'' \in N(v) \setminus \{v_1, v_3\}$. Then $v', v'' \notin V(C)$ by the minimality of C. Hence, $\{v', v''\} \sim \{v_4, v_g\}$ to avoid $(vv', vv'', vv_3v_4, vv_1v_g)$. By symmetry, assume $v' \sim v_g$. By the choice of G, $G - C$ has a 3-coloring c. We can extend c to a 3-coloring of G by letting $c(v_g) = c(v)$ and greedily coloring $v_{g-1}, v_{g-2}, \cdots, v_2, v_1$ in order. This is a contradiction.

Now assume (2) fails and let $x \in N(v_i) \setminus V(C)$ and $y \in N(v_j) \setminus V(C)$ such that $x \sim y$. By the choice of G, $G - C$ has a 3-coloring c with $c(x) \neq c(y)$, contradicting Lemma 3.6.

Suppose (3) fails and, without loss of generality, assume that $G - (C - \{v_1, v_2\})$ has no path from v_1 to v_2 of length 6. Let $w_i \in N(v_i) \setminus V(C)$ for $i = 1, 2$, and let $G' = G - C + w_1w_2$. Then $og(G') \geq 7$. So G' must contain a fork F; otherwise, G' has a 3-coloring which induces a 3-coloring c of $G - C$ with $c(w_1) \neq c(w_2)$, contradicting Lemma 3.6. Since G is fork-free, $w_1w_2 \in E(F)$. If w_1 or w_2 has degree 1 in F, say w_1, then $F - w_1 + \{v_2, w_2v_2\}$ is fork in G, a contradiction. So assume $d_F(w_1) = 4$ and $d_F(w_2) = 2$. Let $w \in F$ with $w \sim w_2$ and $w \neq w_1$. Now $F - \{w, w_2\} + \{v_1, v_2, w_1v_1, v_1v_2\}$ is a fork in G, a contradiction. ∎

4 Excluding shortest cycles of G from $G[V_3]$

The objective of this section is to show that $G[V_3]$ does not contain any shortest odd cycle of G. Along the way we will exclude several reducible configurations from G.

Lemma 4.1. *Let* $C = v_1v_2 \cdots v_7v_1$ *be a shortest odd cycle in* G*, and* $P = v_iu_1 \cdots u_n$ *an induced path such that* $V(P \cap C) = \{v_i\}$*,* $d(u_n, C) \geq 2$*, and* $d(v_i) \geq 4$*. Let* $v \in (N(v_j) \cap N(v_{j+2})) \setminus V(C)$ *for some* $1 \leq j \leq 7$ *(subscripts modulo* 7*). Then* $\{v_j, v_{j+1}, v_{j+2}\} \not\subseteq V_3$*.*

Proof. We choose such P that P is minimal, and we may let $i = 1$. Suppose $\{v_j, v_{j+1}, v_{j+2}\} \subseteq V_3$. Then by Lemma 3.5, $v \notin V_3$. By symmetry, we may assume $j = 4$ or $j = 5$.

Case 1. $j = 5$.

Then $vv_5v_6v_7v$ is a 4-cycle. We claim that $v \notin V(P)$. For, suppose $v = u_s$. Then $s \geq 2$ and $s \neq 3$, since $og(G) \geq 7$. Moreover, u_{s+1} is defined as $d(u_n, C) \geq 2$. If $s \geq 4$ then $(vv_5, vu_{s+1}, vu_{s-1}u_{s-2}, vv_7v_1)$ is a fork in G, a contradiction. Thus $s = 2$. Let $u \in N(u_3) \setminus N(u_1)$ (by Lemma 3.2) such that $u = u_4$ if u_4 is defined. Then $u \notin V(C)$ (as $d(u_n, C) \geq 2$). Note that $v_4 \nsim \{u, u_1\}$ (to avoid C_5); so $v_4 \sim u_3$ to avoid $(vv_7, vu_1, vv_5v_4, vu_3u)$. Hence, $u = u_4$. Let $u' \in N(u_1) \setminus N(v_7)$ (by Lemma 3.2). Then $u' \notin \{v_3, v_4\}$ and $\{u', v\} \nsim v_3$ (to avoid C_5). Hence, $u' \notin V(C)$. Moreover, $u' \notin V(P)$ by the minimality of P. So $u' \sim u_3$ to avoid $(vv_5, vv_7, vu_3u_4, vu_1u')$, and $u_4 \sim v_3$ to avoid $(u_3u_4, u_3u', u_3v_4v_3, u_3vv_7)$. So $n \geq 5$ as $d(u_n, C) \geq 2$. By the minimality of P, $u_5 \nsim \{u', v_4\}$; so $(u_3u', u_3v_4, u_3vv_7, u_3u_4u_5)$ is a fork, a contradiction. Thus we have shown $v \notin V(P)$.

Subcase 1.1. $N(v) \cap V(P) \neq \emptyset$.

We claim that $|N(v)\cap V(P)| = 1$. For, suppose $u_s, u_t \in N(v)$ with $t < s$. Then $s = t+2 \geq 3$ by the minimality of P and the assumption $og(G) \geq 7$. If $t \geq 2$ then $t \geq 3$ (since $og(G) \geq 7$); so $(vv_5, vu_s, vv_7v_1, vu_tu_{t-1})$ is a fork, a contradiction. Thus, $t = 1$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$ as $d(u_n, C) \geq 2$. Since $v_4 \nsim \{u_1, u\}$ (to avoid C_5), $v_4 \sim u_3$ to avoid $(vu_1, vv_7, vv_5v_4, vu_3u)$; so $u = u_4$ (as $d(u_n, C) \geq 2$) and $v_4 \in V_3$ (by the choice of P). Since $v_3 \nsim \{u_2, v\}$ (to avoid C_5), $v_3 \sim u_4$ to avoid $(u_3u_4, u_3u_2, u_3vv_7, u_3v_4v_3)$. Hence, $n \geq 5$ as $d(u_n, C) \geq 2$, and $(u_3v_4, u_3u_2, u_3vv_7, u_3u_4u_5)$ is fork, a contradiction.

Let $u_s \in N(v)$. Then $v_4 \nsim u_{s-1}$ (if $s \geq 2$) to avoid C_5. Let $w_1 \in N(v_1) \setminus \{v_2, v_7, u_1\}$ and $v' \in N(v)\setminus\{v_5, v_7, u_s\}$. Note that $v' \notin V(C)$ by the minimality of C, and if $v' = w_1$ then $s \leq 3$ by the choice of P. Since $og(G) \geq 7$, $w_1 \nsim v_4$ and $v' \nsim \{v_2, v_3\}$. By Lemma 3.2, $v_6 \nsim \{u_s, v'\}$.

Suppose $s \neq 1$. Then $s \geq 3$ since $og(G) \geq 7$. Now $u_s \nsim v_4$; for, otherwise, $v_4 \in V_3$ and $n \geq s + 1$ by the choice of P, and $(u_su_{s+1}, u_sv_4, u_svv_7, u_su_{s-1}u_{s-2})$ would be a fork. Suppose $v' \sim v_1$; then we may assume $w_1 = v'$. Now $w_1 \sim u_{s-1}$ to avoid $(vw_1, vv_7, vv_5v_4, vu_su_{s-1})$, and hence $s = 3$ by the choice of P. Let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$ as $d(u_n, C) \geq 2$, and $u \neq u_1$ since $og(G) \geq 7$. Hence $(vv_7, vw_1, vv_5v_4, vu_3u)$ is a fork, a contradiction. Therefore, $v' \nsim v_1$. Thus $v' \sim u_{s-1}$ to avoid $(vv', vv_5, vv_7v_1, vu_su_{s-1})$, and $v' \sim v_4$ to avoid $(vv', vu_s, vv_7v_1, vv_5v_4)$. Let $u \in N(u_s) \setminus N(v')$. Then $u \notin V(C)$, to avoid C_5 and $N(v_6) \subseteq N(v)$. Hence, $(vv', vv_5, vv_7v_1, vu_su)$ is a fork, a contradiction.

Therefore, $s = 1$. Then, since $og(G) \geq 7$, $u_1 \nsim \{v_3, v_4\}$ and $v_4 \nsim \{u_2, w_1\}$. Moreover, $v' \nsim v_1$ for any choice of v', to avoid $(v_1u_1, v_1v', v_1v_2v_3, v_1v_7v_6)$. Thus $v' \sim \{v_4, u_2\}$ to avoid $(vv', vv_7, vu_1u_2, vv_5v_4)$, and $w_1 \sim \{v_3, v_6\}$ to avoid $(v_1w_1, v_1u_1, v_1v_2v_3, v_1v_7v_6)$. Moreover, $u_2 \sim \{w_1, v_2\}$; otherwise $w_1 \sim v_6$ to avoid $(v_1w_1, v_1v_2, v_1u_1u_2, v_1v_7v_6)$ and $w_1 \nsim v_3$ by the minimality of C, so $(v_1w_1, v_1v_7, v_1u_1u_2, v_1v_2v_3)$ is a fork, a contradiction.

Suppose $u_2 \nsim w_1$. Then $u_2 \sim v_2$; so by the choice of P, $v_2 \in V_3$ and $n \geq 3$. If $v' \sim u_2$ then $v' \nsim v_4$ to avoid C_5 (so $v' \nsim C$); hence with $v'' \in N(v') \setminus N(u_1)$, $(vu_1, vv_7, vv'v'', vv_5v_4)$ would be a fork. Thus $v' \nsim u_2$; so $v' \sim v_4$. Let $v'' \in N(v') \setminus N(v_5)$. Then $v'' \notin V(C)$ (as $og(G) \geq 7$), $v'' \sim u_1$ to avoid $(vv_5, vv_7, vu_1u_2, vv'v'')$, $v'' \sim w_1$ to avoid $(u_1u_2, u_1v'', u_1vv_5, u_1v_1w_1)$, and $\{u_1, w_1\} \nsim v_3$ to avoid C_5. So $w_1 \sim v_6$, and $v'' \sim u_3$ to avoid $(u_1v'', u_1v_1, u_1vv_5, u_1u_2u_3)$. Now $(v''u_3, v''u_1, v''v'v_4, v''w_1v_6)$ is a fork, a contradiction.

Hence, $u_2 \sim w_1$. Then $v' \sim v_4$; otherwise, $v' \sim u_2$, and with $v'' \in N(v') \setminus N(u_1)$, $(vu_1, vv_7, vv'v'', vv_5v_4)$ would be a fork. Also, $v' \sim u_2$. For, suppose $v'' \in N(v') \setminus N(v_5)$. Then $v'' \notin V(C)$, $u_1 \sim v''$ to avoid $(vv_5, vv_7, vv'v'', vu_1u_2)$, $v'' \nsim v_2$ to avoid C_5, and $v_2 \sim u_2$ to avoid $(u_1v'', u_1u_2, u_1vv_5, u_1v_1v_2)$. Hence by the choice of P, $v_2 \in V_3$ and $n \geq 3$. So $v_3 \nsim w_1$ to avoid $N(v_2) \subseteq N(w_1)$, and thus $w_1 \sim v_6$; so $(u_2u_3, u_2v_2, u_2w_1v_6, u_2u_1v)$ is a fork, a contradiction.

Thus, $w_1 \nsim v_3$ and $u_2 \nsim v_2$ to avoid C_5; so $w_1 \sim v_6$. We claim that $v' \in V_3$. For, let $v'' \in N(v') \setminus \{u_2, v, v_4\}$. Then $v'' \notin V(C)$, $v'' \sim w_1$ to avoid $(v'v'', v'v_4, v'vv_7, v'u_2w_1)$, and $v'' \nsim v_3$ to avoid C_5. Now $(v'v'', v'u_2, v'vv_7, v'v_4v_3)$ is a fork, a contradiction.

Moreover, $v_4 \in V_3$. For, let $x \in N(v_4)\setminus\{v', v_3, v_5\}$. Then $x \sim v_2$ to avoid $(v_4x, v_4v', v_4v_5v_6, v_4v_3v_2)$, and $x \nsim u_2$ to avoid C_5. Thus $(v_4x, v_4v_3, v_4v_5v_6, v_4v'u_2)$ is a fork, a contradiction.

If $u_2 \notin V_3$ let $u \in N(u_2) \setminus \{u_1, v', w_1\}$ (with $u = u_3$ if $n \geq 3$), then $u \notin V(C)$, and $(u_2u, u_2u_1, u_2w_1v_6, u_2v'v_4)$ would be a fork. So $u_2 \in V_3$, and hence by Lemma 3.5, $w_1 \notin V_3$. Let $w \in N(w_1) \setminus \{u_2, v_1, v_6\}$. Then $(w_1w, w_1v_1, w_1u_2v', w_1v_6v_5)$ is a fork, a contradiction.

Subcase 1.2. $N(v) \cap V(P) = \emptyset$ for any choice of P.

First, assume $v \nsim w_1$ for every choice of w_1. Let $v', v'' \in N(v) \setminus \{v_5, v_7\}$. Then $v_1 \nsim \{v', v''\}$, $v', v'' \notin V(C \cup P)$, and $\{u_1, v_2, w_1\} \nsim \{v', v''\}$ to avoid C_5. So $v_4 \sim \{v', v''\}$ to avoid $(vv', vv'', vv_7v_1, vv_5v_4)$. Without loss of generality, let $v'' \sim v_4$. Let $v^* \in N(v') \setminus N(v'')$. Then $v^* \notin \{v_2, v_3\}$ to avoid C_5, and $v^* \neq v_6$ to avoid $N(v_6) \subseteq N(v)$; so $v^* \notin V(C)$. Hence $(vv'', vv_5, vv'v^*, vv_7v_1)$ is a fork, a contradiction.

Thus we may choose w_1 so that $v \sim w_1$. Then $w_1 \nsim u_2$, otherwise replacing P with $v_1w_1u_2 \dots u_n$, we get back to Subcase 1.1. Also $w_1 \nsim \{v_3, v_4\}$ to avoid C_5, $w_1 \nsim v_6$ as $N(v_6) \nsubseteq N(v)$, and $|N(u_1) \cap \{v_3, v_6\}| = 1$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$ and by the minimality of C. Thus $u_2 \sim v_2$ to avoid $(v_1w_1, v_1v_7, v_1v_2v_3, v_1u_1u_2)$ or $(v_1w_1, v_1v_2, v_1u_1u_2, v_1v_7v_6)$. Hence by the choice of P, $v_2 \in V_3$ and $n \geq 3$. Let $w \in N(w_1) \setminus N(v_7)$; then $w \notin V(C \cup P)$, and $w \nsim \{u_2, v_3, v_6\}$ to avoid C_5. If $w \nsim u_1$ then, since $|N(u_1) \cap \{v_3, v_6\}| = 1$, $(v_1u_1, v_1v_7, v_1w_1w, v_1v_2v_3)$ or $(v_1u_1, v_1v_2, v_1w_1w, v_1v_7v_6)$ is a fork, a contradiction. Thus $w \sim u_1$, and hence $w \nsim u_3$ (otherwise replacing P with $v_1w_1wu_3 \cdots u_n$ we get back to Subcase 1.1). Again, since $|N(u_1) \cap \{v_3, v_6\}| = 1$, $(u_1v_1, u_1w, u_1v_6v_5, u_1u_2u_3)$ or $(u_1v_3, u_1w, u_1u_2u_3, u_1v_1v_7)$ is a fork, a contradiction.

Case 2. $j = 4$, that is, $vv_4v_5v_6v$ is a 4-cycle.

First, we show $v \notin V(P)$. For, let $v = u_s$. Then $s = 1$ or $s \geq 3$ to avoid C_5, and $n \geq s + 1$ by the choice of P. Let $u \in N(u_{s+1}) \setminus N(u_{s-1})$ such that $u = u_{s+2}$ if $n \geq s+2$. Then $u \notin V(C)$ as $d(u_n, C) \geq 2$. Now $s = 1$; otherwise, $(vv_4, vv_6, vu_{s+1}u, vu_{s-1}u_{s-2})$ would be a fork. Thus, $u_2 \nsim v_2$ to avoid C_5, and $(vv_4, vv_6, vu_2u, vv_1v_2)$ is a fork, a contradiction.

Subcase 2.1. $N(v) \cap V(P) \neq \emptyset$.

Let $u_s \sim v$ with s maximum. Then $s \geq 2$ to avoid C_5. Let $u \in N(u_s) \setminus N(u_{s-2})$ (with $u_0 = v_1$) such that $u = u_{s+1}$ if $n \geq s + 1$. Then $u \notin V(C \cup P)$. Now $N(v) \cap V(P) = \{u_s\}$; for, let $u_t \in N(v)$ with $t < s$, then $2 \leq t \leq s - 2$ (to avoid C_5) and hence $(vv_4, vv_6, vu_su, vu_tu_{t-1})$ (if $t = s - 2$) or $(vv_4, vv_6, vu_su_{s-1}, vu_tu_{t-1})$ (if $t < s - 2$) is a fork, a contradiction.

Let $v' \in N(v) \setminus \{u_s, v_4, v_6\}$; so $v' \notin V(C \cup P)$. Note that $v' \nsim v_3$ or $v' \nsim v_7$ (to avoid C_5). Also note that $u_s \nsim v_3$; for otherwise, $v_3 \in V_3$ and $u = u_{s+1}$ (by the choice of P), and $(u_su_{s+1}, u_sv_3, u_su_{s-1}u_{s-2}, u_svv_6)$ would be a fork.

Suppose $v' \nsim v_3$. Then $v_7 \sim \{v', u_s\}$ to avoid $(vv', vu_s, vv_4v_3, vv_6v_7)$, $v' \sim u_{s-1}$ to avoid $(vv', vv_6, vv_4v_3, vu_su_{s-1})$, and $v' \sim u$ to avoid $(vv', vv_6, vv_4v_3, vu_su)$. Suppose $v_7 \nsim u_s$; so $v_7 \sim v'$, $s = 2$ to avoid $(v'u, v'v_7, v'u_{s-1}u_{s-2}, v'vv_4)$, $u_1 \nsim v_3$ to avoid C_5, $w_1 \sim v_3$ to avoid $(v_1w_1, v_1u_1, v_1v_7v_6, v_1v_2v_3)$, $u_s \nsim \{v_2, w_1\}$ to avoid C_5, and $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_7v_6)$ is a fork, a contradiction. Thus, $v_7 \sim u_s$. Then $v_7 \in V_3$ and $u = u_{s+1}$ by the choice of P, and $s = 2$ to avoid $(u_su_{s+1}, u_sv_7, u_su_{s-1}u_{s-2}, u_svv_4)$. Let $u' \in N(u_3) \setminus N(u_1)$ such that $u' = u_4$ if $n \geq 4$. Then $u' \notin V(C)$, $u' \sim v$ to avoid $(u_2u_1, u_2v_7, u_2vv_4, u_2u_3u')$, and $u' \sim v_3$ to avoid $(vu', vv_6, vv_4v_3, vu_2u_1)$. Hence $v_3 \in V_3$ by the choice of P, and so $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$ is a fork in G, a contradiction.

Hence $v' \sim v_3$; so $v' \nsim \{v_7, u_1\}$ to avoid C_5. Now $v_3 \nsim u_1$; otherwise, replacing P with $v_3u_1u_2 \cdots u_n$, we are back in Case 1. So $v_3 \sim w_1$ to avoid $(v_1w_1, v_1u_1, v_1v_7v_6, v_1v_2v_3)$, and $v' \nsim u_1$ to avoid C_5. Then $w_1 \nsim u_2$; otherwise replacing P with $v_3w_1u_2 \cdots u_n$, we are back in Case 1. Also, $v' \nsim u_i$ for $i \geq 2$; otherwise, $v' \sim u_2$ by the minimality of P, and replacing P with

$v_3v'u_2\dots u_n$, we are back in Case 1 again. Now $u_s \sim v_7$ to avoid $(vv', vv_4, vv_6v_7, vu_su_{s-1})$. Thus, by the choice of P, $v_7 \in V_3$ and $n \geq s+1$. Let $u' \in N(u_{s+1}) \setminus N(u_{s-1})$ such that $u' = u_{s+2}$ if $n \geq s+2$. Then $u' \notin V(C)$, and $(u_su_{s-1}, u_sv_7, u_su_{s+1}u', u_svv_4)$ is a fork in G, a contradiction.

Subcase 2.2. $N(v) \cap V(P) = \emptyset$.

Let $v', v'' \in N(v) \setminus V(C \cup P)$. Then $\{v', v''\} \sim \{v_3, v_7\}$ to avoid $(vv', vv'', vv_4v_3, vv_6v_7)$, $v_1 \nsim \{v', v''\}$ to avoid C_5, $v_5 \nsim \{v', v''\}$ to avoid $N(v_5) \subseteq N(v)$, and $v_3 \sim \{u_1, w_1\}$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$.

Suppose $v_3 \sim u_1$. Then $v_3 \in V_3$, otherwise, replacing P with $v_3u_1u_2\cdots u_n$, we are back in Case 1. Let $v^* \in (N(v') \setminus N(v'')) \cup (N(v'') \setminus N(v'))$ such that $v^* \neq v_7$ (which exists by Lemma 3.2)). By symmetry let $v^* \in N(v') \setminus N(v'')$. Then $v^* \notin V(C)$, and $(vv'', vv_6, vv'v^*, vv_4v_3)$ is a fork, a contradiction.

Hence, $v_3 \nsim u_1$, which implies $v_3 \sim w_1$. Moreover, assume $w_1 \nsim u_2$; for, otherwise, replacing P with $v_1w_1u_2\cdots u_n$, we get back to the situation in the previous paragraph (with $v_3 \sim u_1$). So $u_2 \sim \{v_2, v_7\}$ to avoid $(v_1v_2, v_1w_1, v_1v_7v_6, v_1u_1u_2)$.

If $v_3 \nsim \{v', v''\}$ then as above we may assume that there exists $v^* \in N(v') \setminus N(v'')$ such that $v^* \neq v_7$; so $(vv'', vv_6, vv_4v_3, vv'v^*)$ is a fork, a contradiction. Thus by symmetry, assume $v_3 \sim v'$. Note that $w_1 \nsim v_5$ since $og(G) \geq 7$. If $u_2 \sim v_2$ then $v_2 \in V_3$ (by the choice of P), and $u_2 \sim v'$ to avoid $(v_3w_1, v_3v', v_3v_4v_5, v_3v_2u_2)$; but then, replacing P with $v_3v'u_2\cdots u_n$, we are back in Case 1. So $u_2 \nsim v_2$ and $u_2 \sim v_7$; thus $v_7 \in V_3$ (by the choice of P), and $v_7 \nsim \{v', v''\}$. By Lemma 3.2, let $v^* \in N(v'') \setminus N(v')$. Now $(vv', vv_4, vv''v^*, vv_6v_7)$ is a fork, a contradiction. ∎

Corollary 4.2. *Let $C = v_1\cdots v_7v_1$ be an odd cycle in G and let $v \in V(G) \setminus V(C)$ such that $v \sim v_i$ and $v \sim v_{i+2}$ for $1 \leq i \leq 7$. Then $\{v_i, v_{i+1}, v_{i+2}\} \nsubseteq V_3$.*

Proof. Let $T = \{u \in V(G) \setminus V(C) : d(u, C) \geq 2\}$, and let H denote the subgraph of G obtained by taking the union of all paths P which are from vertices in T to C but internally disjoint from C. Suppose $\{v_i, v_{i+1}, v_{i+2}\} \subseteq V_3$. Then by Lemma 4.1, $V(H) \cap V(C) \subseteq V_3$.

If $T = \emptyset$ then $V(G) = V(C) \cup N_1(C)$, contradicting Corollary 2.2. So $T \neq \emptyset$. Let $K := G \setminus (V(H) \setminus V(C))$. By Lemma 3.7(1), $C \nsubseteq G[V_3]$. So $K \neq C$. Hence by the choice of G, H has a 3-coloring, which induces a 3-coloring c on $C \cap H$. Clearly c can be extended to a 3-coloring of $K[S]$, where $S = V_2(K) \cap V(C)$, in a greedy way. Now applying Corollary 2.2 to K and S, c may be extended further to a 3-coloring of K, and hence a 3-coloring of G, a contradiction. ∎

The next result will be used frequently when we look for a fork in G.

Corollary 4.3. *For any 4-cycle C in G, $|V(C) \cap V_3| \leq 2$.*

Proof. Let $C = v_1v_2v_3v_4v_1$ be a 4-cycle in G such that $v_i \in V_3$ for $1 \leq i \leq 3$. Then $v_4 \notin V_3$ by Lemma 3.5. Let $G' := G/\{v_1, v_3\} - v_2$, and let v denote the identification of v_1 and v_3.

We claim that $og(G') \geq 7$. For, suppose that G' has an odd cycle D with $|V(D)| \leq 5$. Clearly, $v \in V(D)$ and $|V(D)| = 5$, as $og(G) \geq 7$. Let $D = u_1u_2u_3u_4u_5u_1$, with $u_1 = v$. If

$vv_4 \in E(D)$ then assume by symmetry that $v_4 = u_2$ and $v_1u_5 \in E(G)$; so $v_1u_2u_3u_4u_5v_1$ is a C_5 in G, a contradiction. Hence $vv_4 \notin E(D)$. Then by symmetry assume $v_1u_2, v_3u_5 \in E(G)$. Now $v_1u_2 \cdots u_5v_3v_2v_1$ is a 7-cycle, and $v_4v_1v_2v_3v_4$ is a 4-cycle, contradicting Corollary 4.2.

If G' is fork-free then by the choice of G, G' has a 3-coloring which induces a 3-coloring c of $G - \{v_1, v_2, v_3\}$. By coloring v_1 and v_3 with $c(v)$, and v_2 with a color not used by its neighbors, we obtain a 3-coloring of G, a contradiction.

So let F' be a fork in G'. Then $v \in V(F')$, as G is fork-free. If v has degree 1 in F', then let $vx \in E(F')$; by symmetry let $v_1x \in E(G)$, and then $F' - v + \{v_1, v_1x\}$ is a fork in G, a contradiction. So v has degree 2 in F'. Let $vx, vz \in E(F')$, with $v_1x \in E(G)$ (by symmetry). If $v_4 = z$ then $F' - v + \{v_1, v_1x, v_1v_4\}$ is a fork in G, a contradiction. So $v_3z \in E(G)$. By symmetry, assume that z has degree 1 in F'. Let $y \in N(v_2) \setminus \{v_1, v_3\}$. Now, $y \in V(F')$; otherwise $F' - v + \{v_1, v_2, xv_1, v_1v_2\}$ would be a fork in G. So $xy \in E(F')$ to avoid C_5. Similarly, $F' - v$ contains some $v' \in N(v_4) \setminus \{v_1, v_3\}$ and $xv' \in E(F')$; otherwise, $v_4 \notin V(F')$ and $F' - v + \{v_1, v_4, xv_1, v_1v_4\}$ would be a fork in G. Thus $z \nsim \{v', x, y\}$.

Let $G'' := G - \{v_1, v_2, v_3\} + v_4y$. We claim that G'' is fork-free. For, let F'' be a fork in G''. Then $v_4y \in E(F'')$ as G is fork-free. If y is of degree 1 in F'' then $x \notin V(F'')$; so $F'' - y + \{v_1, v_4v_1\}$ is a fork in G, a contradiction. If y is of degree 2 in F'' and let $yy' \in E(F'')$ with $y' \neq v_4$ then, since $x \notin V(F'')$ when $y' \neq x$, $F'' \setminus \{y, y'\} + \{v_1, v_2, v_4v_1, v_1v_2\}$ is a fork in G, a contradiction. Thus y is of degree 4 in F''. Note that $z \notin V(F'')$, since $z \nsim y$ (because of F') and G is C_5-free. If v_4 is of degree 1 in F'' then $F'' - v_4 + \{v_2, yv_2\}$ is a fork in G, a contradiction. So let yv_4v^* be a path in F''. Then $F'' - \{v_4, v^*\} + \{v_2, v_3, yv_2, v_2v_3\}$ is a fork in G, a contradiction.

If $og(G'') \geq 7$ then by the choice of G, G'' has a 3-coloring which induces a 3-coloring of $G'' - v_4y$. By coloring v_2 with $c(v_4)$ and then coloring v_1 and v_3 greedily, we obtain a 3-coloring of G, a contradiction. So $og(G'') \leq 5$. Then, since $og(G) \geq 7$, $og(G'') = 5$ and any 5-cycle in G'' must contain v_4y. Let $D := v_4yy'wv''v_4$ be a 5-cycle in G''.

Since $og(G) \geq 7$, $x \notin V(D)$. So $x \nsim v''$; otherwise $yy'wv''xy$ would be a C_5 in G. Thus, $v' \neq v''$ and, since $og(G) \geq 7$, $v' \notin D$, $v' \nsim y'$ and $z \notin V(D)$. If $y \in V_3(G)$ then we get a contradiction to Corollary 4.2, since in G, $yv_2v_1v_4v''wy'y$ is a 7-cycle, xyv_2v_1x is a 4-cycle, and $\{v_1, v_2, y\} \subseteq V_3(G)$. So $y \notin V_3(G)$. Let $y'' \in N(y) \setminus \{v_2, x, y'\}$. Then $y'' \sim v'$ to avoid $(yy'', yy', yv_2v_3, yxv')$. Now $w \nsim \{x, y''\}$ to avoid C_5. Hence, $(yx, yy'', yv_2v_3, yy'w)$ is a fork, a contradiction. ∎

To prove that $G[V_3]$ contains no shortest odd cycle of G, we need another lemma.

Lemma 4.4. *Let $C := v_1v_2 \cdots v_gv_1$ be a shortest odd cycle in G such that $V(C) \subseteq V_3$, and let $w_i \in N(v_i) \setminus V(C)$ for $1 \leq i \leq g$. Then $w_i \in V_3$ for $1 \leq i \leq g$.*

Proof. Suppose the contrary and, without loss of generality, let $w_1 \notin V_3$. By Corollary 4.2, $N(w_1) \cap V(C) = \{v_1\}$. Let $N(w_1) = \{v_1, x_1, \cdots, x_k\}$; so $k \geq 3$. Now $\{v_2, v_g\} \nsim \{x_1, \ldots, x_k\}$ by Lemma 3.7(2); so $v_1 \nsim N(\{x_1, \cdots, x_k\}) \setminus \{w_1\}$. Thus by Lemma 3.4, there exists $v \in N(\{x_1, \ldots, x_k\}) \setminus \{w_1\}$ such that $|N(v) \cap \{x_1, \ldots, x_k\}| \leq k - 2$. So by symmetry assume $v \nsim \{x_1, x_2\}$ and $v \sim x_3$. Then $(w_1x_1, w_1x_2, w_1v_1v_2, w_1x_3v)$ is a fork, a contradiction. ∎

Corollary 4.5. *If $V(C) = v_1 \cdots v_gv_1$ is a shortest odd cycle, then $V(C) \nsubseteq V_3$.*

Proof. Suppose $V(C) \subseteq V_3$. Let $w_i \in N(v_i) \setminus V(C)$ for $i = 1, \ldots, g$. Then $w_i \in V_3$ by Lemma 4.4. By Lemma 3.7(3), there is a path $v_1w_1xyzw_2v_2$ internally disjoint from C. We may choose this path so that $|\{x, z\} \cap V_3|$ is minimum.

Suppose $x, z \in V_3$. Then by Lemma 3.2, $y \notin V_3$. So let $y_1, y_2 \in N(y)\backslash\{x, z\}$. Then $\{y_1, y_2\} \sim \{w_1, w_2\}$ to avoid $(yy_1, yy_2, yxw_1, yzw_2)$. By symmetry, we may assume $y_1 \sim w_1$. So $y_1 \notin \{v_i, w_i : 1 \leq i \leq n\}$ (by Lemma 3.7(2)). By Corollary 4.3, $y_1 \notin V_3$ (because of the 4-cycle $y_1yxw_1y_1$). Thus, the path $w_1y_1yzw_2$ contradicts the choice of w_1xyzw_2.

So by symmetry, we may assume $x \notin V_3$. Let $N(x) \setminus \{w_1\} = \{x_1, \ldots, x_k\}$; then $k \geq 3$. There exists $y_1 \in N(w_1) \cap (N(\{x_1, \ldots, x_k\}) \setminus \{x\})$ such that $|N(y_1) \cap \{x_1, \ldots, x_k\}| = 1$. For, otherwise, by Lemma 3.4, for some $t \in N(\{x_1, \ldots, x_k\}) \setminus N(w_1)$, $|N(t) \cap \{x_1, \ldots, x_k\}| \leq k-2$, and let $t \sim x_1$ and $t \nsim \{x_2, x_3\}$. Now $(xx_2, xx_3, xx_1t, xw_1v_1)$ is a fork, a contradiction.

Let $y = x_k$. Now $z \sim \{x_1, x_2\}$ to avoid $(xx_1, xx_2, xw_1v_1, xyz)$. Thus, by the symmetry between y and $N(z) \cap \{x_1, x_2\}$, we may assume $y_1 \sim x_1$; so $y_1 \nsim N(x)\backslash\{w_1, x_1\}$. Let $u \in N(x_2)\backslash N(y)$ and $v \in N(y)\backslash N(x_2)$. Then $u, v \notin \{v_1, v_2, w_1, w_2, x, y, y_1\}$, $u \neq z$, $x_1 \sim u$ to avoid $(xx_1, xy, xw_1v_1, xx_2u)$, and $x_1 \sim v$ to avoid $(xx_1, xx_2, xw_1v_1, xyv)$.

Case 1. $v = z$ for every $v \in N(y) \setminus N(x_2)$ and for every choice of w_1xyzw_2.

First, assume $y \notin V_3$, and let $y', y'' \in N(y) \setminus \{x, z\}$. Then $w_2 \sim \{y', y''\}$ to avoid $(yy', yy'', yxw_1, yzw_2)$, and assume $w_2 \sim y'$ by symmetry. Then $z \notin V_3$, for otherwise, $y' \notin V_3$ by Corollary 4.3, and $w_1xyy'w_2$ contradicts the choice w_1xyzw_2. Let $z' \in N(z) \setminus \{w_2, x_1, y\}$. Note that $z' \notin V(C)$ as $z \neq w_i$ for all i (by Lemma 3.7(2)), and thus $z' \nsim v_2$. Now $z' \sim y_1$ to avoid $(zz', zy, zw_2v_2, zx_1y_1)$ and $z' \sim u$ to avoid $(zz', zy, zw_2v_2, zx_1u)$. So $z' \nsim y''$ to avoid $(z'y'', z'u, z'y_1w_1, z'zw_2)$, and $x_1 \sim y''$ to avoid $(zz', zx_1, zw_2v_2, zyy'')$. Now $(x_1y'', x_1u, x_1xw_1, x_1zw_2)$ is a fork in G, a contradiction.

Thus, $y \in V_3$. Let $y' \in N(y) \setminus \{x, z\}$. Then $y' \sim x_2$ by the definition of v, $y' \nsim x_1$ to avoid $N(y) \subseteq N(x_1)$, and $x_2 \nsim z$ to avoid $N(y) \subseteq N(x_2)$. So $x \in V_4$. For, if $x \notin V_4$ then $x_3 \sim z$ to avoid $(xx_3, xx_2, xw_1v_1, xyz)$, $x_3 \nsim y'$ to avoid $N(y) \subseteq N(x_3)$, and $y' \sim w_2$ to avoid $(zx_1, zx_3, zyy', zw_2v_2)$. Thus $(zy, zx_3, zx_1y_1, zw_2v_2)$ is a fork, a contradiction.

Moreover, $x_2 \in V_3$. For, let $x_2' \in N(x_2) \setminus \{u, x, y'\}$. Then $x_2' \sim x_1$ to avoid $(xx_1, xy, xw_1v_1, xx_2x_2')$, and $w_2 \sim \{u, x_2'\}$ to avoid $(x_1u, x_1x_2', x_1xw_1, x_1zw_2)$. If $w_2 \sim u$ then $(x_2x_2', x_2y', x_2xw_1, x_2uw_2)$ is a fork, and if $w_2 \sim x_2'$ then $(x_2y', x_2u, x_2xw_1, x_2x_2'w_2)$ is a fork, a contradiction.

Now $x_1 \in V_4$. For, assume $x_1' \in N(x_1) \setminus \{u, x, y_1, z\}$. Then $w_2 \sim \{u, x_1'\}$ to avoid $(x_1x_1', x_1u, x_1xw_1, x_1zw_2)$. If $w_2 \sim u$ then $(x_1x_1', x_1y_1, x_1xy, x_1uw_2)$ is a fork, and if $w_2 \sim x_1'$ then $(x_1u, x_1y_1, x_1xy, x_1x_1'w_2)$ is a fork, a contradiction.

Next, assume $y_1 \notin V_3$, and let $y_1', y_1'' \in N(y_1) \setminus \{w_1, x_1\}$. Thus $z \sim \{y_1', y_1''\}$ to avoid $(y_1y_1', y_1y_1'', y_1w_1v_1, y_1x_1z)$, and let $z \sim y_1'$. By applying the above argument for x and $v_1w_1xyzw_2v_2$ to z and $v_2w_2zyxw_1v_1$, we have $z \in V_4$, $|N(z) \cap V_3| = 3$ (so $y_1' \in V_3$), and $N(w_2) \cap (N(x_1) \setminus \{w_2\})| = \{u\}$ (as $w_2 \nsim \{y_1, y_1'\}$ since $og(G) \geq 7$). So $u \nsim y_1'$ to avoid $N(y_1') \subseteq N(x_1)$. Hence $u \sim y_1''$ to avoid $(y_1y_1'', y_1y_1', y_1w_1v_1, y_1x_1u)$. Now $N(y_1'') = \{u, y_1, y'\}$; otherwise let $y_1^* \in N(y_1'') \setminus \{u, y_1, y'\}$, then $(ux_1, ux_2, uy_1''y_1^*, uw_2v_2)$ is a fork, a contradiction. Thus $y_1 \in V_4$ by Lemma 3.2. Note that $y' \sim y_1'$ to avoid $(zy_1', zx_1, zw_2v_2, zyy')$. So $y' \in V_4$; otherwise let $y^* \in N(y') \setminus \{y, y_1', y_1''\}$ then $(y'y^*, y'y_1'', y'y_1'z, y'x_2x)$ is a fork, a contradiction. Also, $u \in V_4$; otherwise, let $u^* \in N(u) \setminus \{w_2, x_1, x_2, y_1''\}$ then $(uu^*, uy_1'', ux_1x, uw_2v_2)$ is a fork, a contradiction. Therefore, $\{v_1w_1, v_2w_2\}$ is an edge-cut of G, which implies that in

$G-(C-\{w_2,w_3\})$ there is no path from w_2 to w_3 of length 6, contradicting Lemma 3.7(3).

Thus, $y_1 \in V_3$. Let $G' := G/\{x_2,y\} - \{w_1,x,x_1,y_1\}$. Using the path ux_1z, we can show that G' is fork-free and $og(G') \geq 7$. So by the choice of G, G' has a 3-coloring which induces a 3-coloring c of $G-\{w_1,x,x_1,y_1\}$ with $c(x_2)=c(y)$. Now coloring x_1 with $c(y)$, and greedily coloring y_1,w,x in order, we get a 3-coloring of G, a contradiction.

Case 2. $v \neq z$ for some choice of v and w_1xyzw_2, and $x_1 \sim z$.

Let $y' \in N(y)\backslash N(x_1)$ (by Lemma 3.2. Note that $y' \notin \{v_1,v_2,w_1,w_2,x,y,z,x_1,x_2,u,y_1\}$. Hence $w_2 \sim \{v,y'\}$ to avoid (yv,yy',yxw_1,yzw_2), and $w_2 \sim \{u,v\}$ to avoid $(x_1u,x_1v,x_1xw_1,x_1zw_2)$. Therefore, since $w_2 \in V_3$, $w_2 \sim v$ and $w_2 \nsim \{y',u\}$. Then $y' \nsim x_1$ to avoid $(x_1u,x_1y',x_1xw_1,xvw_2)$, $y' \sim x_2$ to avoid (xx_1,xx_2,xw_1v_1,xyy'), and $z \nsim x_2$ to avoid $(x_2u,x_2y',x_2xw_1,x_2zw_2)$.

Let $w \in N(y_1)\backslash N(x)$. Then $w \nsim x_2$ to avoid C_5, $w \sim \{v,z\}$ to avoid $(x_1v,x_1z,x_1xx_2,x_1y_1w)$, and $w \sim \{u,z\}$ to avoid $(x_1u,x_1x,x_1zw_2,x_1y_1w)$. If $w \nsim z$ then $w \sim u$ and $w \sim v$, and $y' \sim w$ to avoid (vw,vx_1,vyy',vw_2v_2); so (wu,wy',wvw_2,wy_1w_1) is a fork, a contradiction. Hence, $w \sim z$. Then $w \sim u$ to avoid (zy,zw,zx_1u,zw_2v_2), and $w \sim y'$ to avoid (zx_1,zw,zw_2v_2,zyy'). Thus (wu,wy',wzw_2,wy_1w_1) is a fork in G, a contradiction.

Case 3. $v \neq z$ for some choice of v and w_1xyzw_2, and $x_1 \nsim z$.

So $x_2 \sim z$ to avoid (xx_1,xx_2,xw_1v_1,xyz). Thus x_2 and y are symmetric. We claim that $N(x_2)\setminus\{u\} = N(y)\setminus\{v\}$. For, otherwise, by symmetry, let $y' \in N(y)\setminus\{v\}\setminus(N(x_2)\setminus\{u\})$. Then $y' \sim x_1$ to avoid (xx_1,xx_2,xw_1v_1,xyy'), and $w_2 \sim \{y',v\}$ to avoid (yv,yy',yxw_1,yzw_2). By symmetry between v and y', assume $y' \sim w_2$. Then $(x_1u,x_1v,x_1xw_1,x_1y'w_2)$ is a fork in G, a contradiction.

Therefore, by Lemma 3.3, there exist $u' \in N(u)\backslash\{x_2\}\backslash(N(v)\backslash\{y\})$ and $v' \in N(v)\backslash\{y\}\backslash(N(u)\backslash\{x_2\})$. Note that $w_1 \notin \{u',v'\}$ as $w_1 \in V_3$, $v' \nsim x_2$ and $u' \nsim y$ to avoid C_5, $v' \nsim x$ to avoid (xy,xv',xx_2u,xw_1v_1), and $u' \nsim x$ to avoid (xx_2,xu',xyv,xw_1v_1). So $y_1 \sim \{u',v'\}$ to avoid $(x_1y_1,x_1x,x_1uu',x_1vv')$. By symmetry, assume $y_1 \sim u'$. By Corollary 4.3, $\{x_2,y,z\} \nsubseteq V_3$.

Suppose $x_2 \notin V_3$. Let $w \in N(x_2)\backslash\{x,z\}$; then $w \sim y$. If $w_2 \nsim w$ then, since $w_2 \nsim u$ or $w_2 \nsim v$ (as $w_2 \in V_3$), (yv,yw,yxw_1,yzw_2) or $(x_2u,x_2w,x_2xw_1,x_2zw_2)$ is a fork in G, a contradiction. So $w_2 \sim w$, and we have symmetry between w and z. Note that $u' \sim \{w,z\}$ to avoid $(x_2w,x_2z,x_2xw_1,x_2uu')$. So $(wx_2,wy,wu'y_1,ww_2v_2)$ (when $u' \sim w$) or $(zx_2,zy,zu'y_1,zw_2v_2)$ (when $u' \sim z$) is a fork in G, a contradiction.

Hence, $x_2 \in V_3$; so $y \in V_3$, and hence $z \notin V_3$ (by Corollary 4.3). Now $x \in V_4$; otherwise $x_3 \sim z$ to avoid (xx_1,xx_3,xw_1v_1,xyz), and $x_3 \sim u$ to avoid (xy,xx_3,xw_1v_1,xx_2u), which implies $N(x_2) \subseteq N(x_3)$, contradicting Lemma 3.2. Then $z \in V_4$; for otherwise let $z',z'' \in N(z)\setminus\{w_2,x_2,y\}$, then (zz',zz'',zw_2v_2,zyx) is a fork, a contradiction.

By the choice of G, $G-\{v_1,v_2,w_1,w_2,x,x_2,y,z\}$ has a 3-coloring c. By letting $c(x_2)=c(y)=c(x_1)$ and greedily coloring z,w_2,v_2,v_1,w_1,x in this order, we obtain a 3-coloring of G, a contradiction. ∎

5 Structure around a shortest odd cycle

We derive certain useful properties about the structure of G around a shortest odd cycle.

Lemma 5.1. *Let $C := v_1 \cdots v_g v_1$ be a shortest odd cycle in G. Then there exist $z \notin V(C) \cup N_1(C)$ and a path Z from z to some v_i such that $V(Z) \cap V(C) = \{v_i\}$ and $d(v_i) \geq 4$.*

Proof. Let $T := V(G) \setminus (V(C) \cup N_1(C))$, and let H denote the union of all paths P which are from vertices in T to C but internally disjoint from C. Then $T \neq \emptyset$; otherwise $V(G) = V(C) \cup N_1(C)$ and so $\chi(G) \leq 3$ by Corollary 2.2, a contradiction.

Now suppose that the assertion of the lemma fails. Then $V(H) \cap V(C) \subseteq V_3$. Let $K := G \setminus (V(H) \setminus V(C))$. By Corollary 4.5, $V(C) \not\subseteq V(H)$. So $K \setminus V(C) \neq \emptyset$. Thus by the choice of G, H has a 3-coloring, say c. Clearly c can be extended to a 3-coloring of $K[S]$, where $S = V_2(K) \cap V(C)$. Now applying Corollary 2.2 to K and S, c can be extended to a 3-coloring of $H \cup K = G$, a contradiction. ∎

We choose a triple (C, P, v_i) satisfying Lemma 5.1 so that

$$|V(C) \cap V_3| \text{ is maximum, and then } P \text{ is minimal while fixing } z.$$

Without loss of generality, assume $i = 1$, and let $P = v_1 u_1 \ldots u_n$, with $z = u_n \notin V(C) \cup N_1(C)$ (i.e., $d(u_n, C) \geq 2$). We will show that $n \leq 2$. Let $w_1 \in N(v_1) \setminus \{u_1, v_g, v_2\}$. By the minimality of P, P is an induced path.

Lemma 5.2. *If $u_1 \sim v_{g-1}$ then $v_g \nsim P - v_1$, and if $u_1 \sim v_3$ then $v_2 \nsim P - v_1$.*

Proof. Suppose the assertion is false. By symmetry, assume $u_1 \sim v_{g-1}$ and $v_g \sim u_s$ for some $1 \leq s \leq n$. Then $s \geq 4$, since $s \notin \{1, 3\}$ (as $og(G) \geq 7$) and $s \neq 2$ to avoid $N(v_g) \subseteq N(u_1)$. Since $d(u_n, C) \geq 2$, $n \geq s + 1$. By the choice of (C, P, v_1), $v_g \in V_3$, $v_{g-1} \nsim P - u_1$, and $w_1 \nsim P - \{u_2, v_1\}$.

Moreover, $u_s \nsim v_2$; for otherwise, $v_2 \in V_3$ by the choice of (C, P, v_1), and hence $(u_s u_{s+1}, u_s v_2, u_s v_g v_{g-1}, u_s u_{s-1} u_{s-2})$ would be a fork in G. Hence, $v_3 \sim w_1$ to avoid $(v_1 u_1, v_1 w_1, v_1 v_g u_s, v_1 v_2 v_3)$, and $u_2 \sim \{w_1, v_2\}$ to avoid $(v_1 w_1, v_1 v_2, v_1 u_1 u_2, v_1 v_g u_s)$.

Case 1. $u_2 \sim w_1$.

Let $v \in N(v_2) \setminus N(w_1)$. Then $v \notin V(C)$ by the choice of (C, P, v_1). If $v \in V(P)$ then $v \neq u_s$ (as $u_s \nsim v_2$), $v \notin \{u_{s-1}, u_{s+1}\}$ (to avoid C_5), $v \neq u_1$ (to avoid C_3), $v \neq u_2$ (otherwise $v_2 \in V_3$ and $N(v_2) \subseteq N(w_1)$); so $(v_1 w_1, v_1 u_1, v_1 v_g u_s, v_1 v_2 v)$ is a fork in G, a contradiction. Hence, $v \notin V(P)$. Now $v \sim u_1$ to avoid $(v_1 u_1, v_1 w_1, v_1 v_g u_s, v_1 v_2 v)$, and $\{v_4, v_{g-2}\} \nsim \{u_2, v\}$ and $v_3 \nsim u_3$ by the choice of (C, P, v_1). So $v \sim u_3$ to avoid $(u_1 v, u_1 v_1, u_1 v_{g-1} v_{g-2}, u_1 u_2 u_3)$. Hence, $v_2 \in V_3$ by the choice of (C, P, v_1). Therefore, $|\{w_1, v_3\} \cap V_3| \leq 1$ by Corollary 4.3.

Suppose $w_1 \notin V_3$. Let $w \in N(w_1) \setminus \{u_2, v_1, v_3\}$. Clearly, $w \notin V(C \cup P)$ by the choice of (C, P, v_1), and $w \neq v$ as $v \nsim w_1$. So $w \sim u_3$ to avoid $(w_1 w, w_1 v_3, w_1 v_1 v_g, w_1 u_2 u_3)$, and $w \sim v_4$ to avoid $(w_1 w, w_1 u_2, w_1 v_1 v_g, w_1 v_3 v_4)$. Therefore $v_4 \in V_3$ by the choice of (C, P, v_1); so $(u_3 u_4, u_3 u_2, u_3 w v_4, u_3 v v_2)$ is a fork in G, a contradiction.

Thus, $w_1 \in V_3$, and $v_3 \notin V_3$. Let $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $y \notin V(C \cup P)$ and $y \nsim P - \{u_1, v_1\}$ by the choice of (C, P, v_1), and $y \neq v$ as $og(G) \geq 7$. If $y \sim v_5$ then $y \nsim \{v, u_2\}$ by the choice of (C, P, v_1); so $(v_3 y, v_3 v_4, v_3 v_2 v, v_3 w_1 u_2)$ is a fork in G, a contradiction. So $y \nsim v_5$, and hence $y \sim u_2$ to avoid $(v_3 y, v_3 v_2, v_3 v_4 v_5, v_3 w_1 u_2)$. Now $(u_2 y, u_2 w_1, u_2 u_3 u_4, u_2 u_1 v_{g-1})$ is a fork in G, a contradiction.

Case 2. $u_2 \nsim w_1$.

Hence, $w_1 \nsim P - v_1$, $u_2 \sim v_2$, and $v_2 \in V_3$ (by the choice of (C, P, v_1)). If $w_1 \in V_3$ then $C' := v_1 w_1 v_3 \cdots v_1$ is a shortest odd cycle in G such that $|V(C') \cap V_3| = |V(C) \cap V_3|$; so (C', P, v_1) is a triple that gives the situation in Case 1. Hence, we may assume $w_1 \notin V_3$.

For any $w \in N(w_1) \backslash \{v_1, v_3\}$, $w \notin V(C) \cup V(P - \{u_1, u_2\})$ by the choice of (C, P, v_1)), $w \nsim u_2$ as $w_1 \nsim u_2$, and $w_1 \nsim u_1$ to avoid triangle; so $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gu_s, v_1w_1w)$, and hence $v_4 \nsim w$ by the choice of (C, P, v_1). Let $w', w'' \in N(w_1) \setminus \{v_1, v_3\}$ be distinct; then $(w_1w', w_1w'', w_1v_3v_4, w_1v_1v_g)$ is a fork in G, a contradiction. ∎

Lemma 5.3. *If $w_1 \sim v_3$ then $v_2 \nsim P - v_1$ unless $u_1 \nsim \{v_3, v_{g-1}\}$, $v_2 \sim u_2$, $\{v_2, w_1\} \subseteq V_3$, and $w_1 \nsim P - v_1$; and if $w_1 \sim v_{g-1}$ then $v_g \nsim P - v_1$ unless $u_1 \nsim \{v_3, v_{g-1}\}$, $v_g \sim u_2$, $\{v_g, w_1\} \subseteq V_3$, and $w_1 \nsim P - v_1$.*

Proof. By symmetry we only prove the first half of the statement. Let $w_1 \sim v_3$ and $v_2 \sim u_s$ for some $1 \leq s \leq n$. Then $s \geq 2$ to avoid C_3, and $s \neq 3$ to avoid C_5. Moreover, by the choice of (C, P, v_1), $v_2 \in V_3$ and $n \geq s + 1$. Hence, $w_1 \nsim u_s$ to avoid $N(v_2) \subseteq N(w_1)$. By Lemma 5.2, $w_1 \nsim u_2$ and $u_1 \nsim v_3$. Thus, since $og(G) \geq 7$, $\{w_1, v_3\} \nsim P - v_1$ by the choice of (C, P, v_1).

We claim that $s = 2$. For, suppose $s \geq 4$. Then $u_s \nsim v_g$; otherwise, $v_g \in V_3$ by the choice of (C, P, v_1), and $(u_su_{s+1}, u_sv_g, u_su_{s-1}u_{s-2}, u_sv_2v_3)$ would be a fork in G. Also, $u_s \nsim v_{g-1}$ to avoid C_5. Hence $u_1 \sim v_{g-1}$ to avoid $(v_1w_1, v_1u_1, v_1v_2u_s, v_1v_gv_{g-1})$. Therefore, $v_g \nsim P - v_1$ by Lemma 5.2. So $(v_1v_g, v_1w_1, v_1u_1u_2, v_1v_2u_s)$ is a fork in G, a contradiction.

Case 1. $u_1 \sim v_{g-1}$. By Corollary 4.3, $\{u_1, u_2\} \nsubseteq V_3$.

Subcase 1.1. $u_2 \in V_3$.

Thus $u_1 \notin V_3$, and let $u \in N(u_1) \setminus \{u_2, v_1, v_{g-1}\}$. Then $u \notin V(C \cup P)$ and $v_g \in V_3$ (by the choices of (C, P, v_1)), and $u \nsim v_g$ to avoid $N(v_g) \subseteq N(u_1)$. Let $N(v_g) \setminus N(u_1) = \{v\}$. So $v \notin V(C)$ by the choice of (C, P, v_1), and $v \notin V(P)$ by Lemma 5.2. Then $w_1 \sim \{u, v\}$ to avoid $(v_1w_1, v_1v_2, v_1u_1u, v_1v_gv)$, $u \sim \{u_3, v_{g-2}\}$ to avoid $(u_1u, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$, $u \sim \{u_3, w_1\}$ to avoid $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$, and $u \sim \{w_1, v_{g-2}\}$ to avoid $(u_1u, u_1u_2, u_1v_{g-1}v_{g-2}, u_1v_1w_1)$. But $u \nsim w_1$ or $u \nsim v_{g-2}$ (by the choice of (C, P, v_1)); so $u \sim u_3$.

Suppose $u \nsim w_1$ and $u \sim v_{g-2}$. Then $v_{g-2} \in V_3$ and $u_3 \nsim C$ by the choice of (C, P, v_1). If $u_3 \notin V_3$, then let $u', u'' \in N(u_3) \setminus \{u, u_2\}$; now $(u_3u', u_3u'', u_3uv_{g-2}, u_3u_2v_2)$ is a fork, a contradiction. So $u_3 \in V_3$, and hence $u \notin V_3$ by Corollary 4.3. Let $u' \in N(u) \setminus \{u_1, u_3, v_{g-2}\}$. Then $u' \neq v$ to avoid C_5. By the choice of (C, P, v_1), $u' \notin V(C \cup P)$ and $u' \neq w_1$, $u_3 \nsim v_{g-3}$, and $u_1 \nsim \{v_{g-2}, v_{g-3}\}$. So $u' \sim \{v_{g-1}, v_1\}$ to avoid $(u_1u_2, u_1v_{g-1}, u_1v_1w_1, u_1uu')$. If $u' \nsim v_{g-1}$ then $u' \sim v_1$; so $(v_1u', v_1w_1, v_1v_2u_2, v_1v_gv_{g-1})$ would be a fork. Hence, $u' \sim v_{g-1}$. Then $u' \sim v$ to avoid $(v_{g-1}v_{g-2}, v_{g-1}u', v_{g-1}u_1u_2, v_{g-1}v_gv)$, and $u' \sim v_{g-3}$ to avoid $(v_{g-1}u', v_{g-1}v_g, v_{g-1}u_1u_2, v_{g-1}v_{g-2}v_{g-3})$. Hence $(u'v, u'v_{g-1}, u'v_{g-3}v_{g-4}, u'uu_3)$ is a fork, a contradiction.

Therefore, $u \sim w_1$ and $u \nsim v_{g-2}$. First, assume $u_3 \notin V_3$, and let $u', u'' \in N(u_3) \setminus \{u, u_2\}$ be distinct. Then $w_1 \sim \{u', u''\}$ to avoid $(u_3u', u_3u'', u_3u_2v_2, u_3uw_1)$, and $v \nsim u_3$ to avoid $(u_3u', u_3u, u_3u_2v_2, u_3vv_g)$ or $(u_3u'', u_3u, u_3u_2v_2, u_3vv_g)$. Without loss of generality, let $w_1 \sim u'$. Note that $v_{g-2} \nsim \{u, u'\}$ and $u \nsim v_4$ by the choice of (C, P, v_1), and $u_1 \sim u'$ or $v \sim w_1$ to avoid $(v_1u_1, v_1v_2, v_1w_1u', v_1v_gv)$. If $u' \sim u_1$ then $(u_1u, u_1u', u_1v_1v_2, u_1v_{g-1}v_{g-2})$ would be a fork. So

$v \sim w_1$. Then $(w_1v, w_1v_1, w_1v_3v_4, w_1uu_3)$ is a fork in G, a contradiction.

Hence, $u_3 \in V_3$; so $u \notin V_3$ by Corollary 4.3. Let $u' \in N(u) \setminus \{u_1, u_3, w_1\}$. Then $u' \neq v$ and $u' \nsim v_{g-2}$ to avoid C_5, $u' \nsim v_{g-3}$ by the choice of (C, P, v_1), and $u' \sim \{v_{g-1}, v_1\}$ to avoid $(u_1u_2, u_1v_1, u_1v_{g-1}v_{g-2}, u_1uu')$. If $u' \sim v_{g-1}$ then $(v_{g-1}u', v_{g-1}v_g, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}u_1u_2)$ would be a fork. So $u' \nsim v_{g-1}$, and $u' \sim v_1$. Then $(v_1u', v_1w_1, v_1v_2u_2, v_1v_gv_{g-1})$ is a fork in G, a contradiction.

Subcase 1.2. $u_2 \notin V_3$.

Let $u \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u \notin V(C)$ (by the choice of (C, P, v_1) and because of $u_2 \nsim v_g$ by Lemma 5.2), $u \neq w_1$ as $u_2 \nsim w_1$, and $u \nsim w_1$ to avoid C_5. Let $u', u'' \in N(u_3) \setminus \{u_2\}$ such that $u' \notin N(u_1)$ and $u'' \notin N(u)$. Note that u', u'' need not be distinct, $\{u', u''\} \cap \{v_4, v_{g-2}\} = \emptyset$ to avoid C_5, and so $u', u'' \notin V(C)$ by the choice of (C, P, v_1) and Lemma 5.2.

Then $u \sim \{v_{g-1}, u'\}$ to avoid $(u_2u, u_2v_2, u_2u_3u', u_2u_1v_{g-1})$, $u \sim v_{g-1}$ or $u'' \sim u_1$ to avoid $(u_2u, u_2v_2, u_2u_3u'', u_2u_1v_{g-1})$, and $u \sim \{v_{g-1}, v_3\}$ to avoid $(u_2u, u_2u_3, u_2u_1v_{g-1}, u_2v_2v_3)$.

Suppose $u \nsim v_{g-1}$. Then $u'' \sim u_1$, and $u \nsim v_1$ to avoid $N(v_2) \subseteq N(u)$. By the choice of (C, P, v_1), $u'' \nsim w_1$ or $u'' \nsim v_{g-2}$. Now $(u_1v_1, u_1u'', u_1v_{g-1}v_{g-2}, u_1u_2u)$ (when $u'' \nsim v_{g-2}$) or $(u_1u'', u_1v_{g-1}, u_1u_2u, u_1v_1w_1)$ (when $u'' \nsim w_1$) is a fork, a contradiction.

Therefore $u \sim v_{g-1}$. So $u \nsim v_3$ by the minimality of C, $u \sim u'$ to avoid $(u_2u, u_2u_1, u_2v_2v_3, u_2u_3u')$, and $u_1 \sim u''$ to avoid $(u_2u, u_2u_1, u_2v_2v_3, u_2u_3u'')$. Note that $v_g \nsim u''$ to avoid $N(v_g) \subseteq N(u_1)$, and $u \nsim v_{g-3}$ by the choice of (C, P, v_1). Therefore, $u'' \sim v_{g-2}$ to avoid $(v_{g-1}v_g, v_{g-1}u, v_{g-1}u_1u'', v_{g-1}v_{g-2}v_{g-3})$; so $v_{g-2} \in V_3$ and $u'' \nsim w_1$ by the choice of (C, P, v_1). Further, $u' \sim v_g$ to avoid $(v_{g-1}u_1, v_{g-1}v_g, v_{g-1}uu', v_{g-1}v_{g-2}v_{g-3})$. So $v_g \in V_3$ by the choice of (C, P, v_1). Hence, $(v_1w_1, v_1v_2, v_1v_gu', v_1u_1u'')$ (if $u' \nsim w_1$) or $(u'u, u'v_g, u'u_3u'', u'w_1v_3)$ (if $u' \sim w_1$) is a fork, a contradiction.

Case 2. $u_1 \nsim v_{g-1}$.

Then $w_1 \notin V_3$; otherwise we would have the exceptional case. Let $w, w' \in N(w_1) \setminus \{v_1, v_3\}$. Suppose $\{w, w'\} \nsim v_g$. Then $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$ and $w' \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w')$. Note $v_g \nsim u_3$ to avoid C_5. If $v_g \sim u_2$ then $v_g \in V_3$ by the choice of (C, P, v_1), and hence $u_3 \sim w$ to avoid $(u_2u_3, u_2v_g, u_2v_2v_3, u_2u_1w)$; if $v_g \nsim u_2$ then $u_3 \sim \{w, w'\}$ to avoid $(u_1w, u_1w', u_1u_2u_3, u_1v_1v_g)$. Hence, by symmetry, assume $u_3 \sim w$. Then $w' \sim u_3$ to avoid $(w_1w', w_1v_3, w_1v_1v_g, w_1wu_3)$, and $v_4 \sim \{w, w'\}$ to avoid $(w_1w, w_1w', w_1v_1v_g, w_1v_3v_4)$. By symmetry assume $w' \sim v_4$. Then $v_4 \in V_3$ and $u_3 \nsim v_{g-1}$ by the choice of (C, P, v_1). If $u_2 \nsim v_g$ then $(u_1u_2, u_1w, u_1w'v_4, u_1v_1v_g)$ would be a fork. So $u_2 \sim v_g$; and hence $(u_2u_3, u_2u_1, u_2v_2v_3, u_2v_gv_{g-1})$ is a fork, a contradiction.

Thus, $\{w, w'\} \sim v_g$, and assume $w \sim v_g$ by symmetry.

Subcase 2.1. There exists $u \in N(u_1) \setminus (N(w_1) \cup \{u_2\})$.

Then $u \neq v_{g-1}$ (as $u_1 \nsim v_{g-1}$); so $u \notin V(C \cup P)$ by the choice of (C, P, v_1). Now $u \sim v_g$ to avoid $(v_1w_1, v_1v_2, v_1u_1u, v_1v_gv_{g-1})$, $u \sim v_{g-2}$ to avoid $(v_gu, v_gw, v_gv_1v_2, v_gv_{g-1}v_{g-2})$, and $\{u, w\} \nsim u_3$ and $u_2 \nsim v_4$ by the choice of (C, P, v_1).

Suppose $u_1 \notin V_3$, and let $u' \in N(u_1) \setminus \{u, u_2, v_1\}$. Then $u' \notin V(C \cup P) \cup \{w_1\}$ and $u' \nsim v_4$ by the choice of (C, P, v_1), $u' \sim \{u_3, w_1\}$ to avoid $(u_1u', u_1u, u_1u_2u_3, u_1v_1w_1)$, and $u' \sim \{u_3, v_{g-2}\}$ to avoid $(u_1u', u_1v_1, u_1u_2u_3, u_1uv_{g-2})$. By the choice of (C, P, v_1), $u' \nsim w_1$ or $u' \nsim v_{g-2}$; so

$u' \sim u_3$. Hence, again by the choice of (C, P, v_1), $u' \nsim v_{g-2}$ and $u' \neq w$. If $u' \nsim w_1$ then $(u_1u', u_1u_2, u_1uv_{g-2}, u_1v_1w_1)$ would be a fork. So $u' \sim w_1$, and $(w_1w, w_1v_1, w_1u'u_3, w_1v_3v_4)$ is a fork, a contradiction.

Hence, $u_1 \in V_3$. By Corollary 4.3, $u_2 \notin V_3$, and let $u' \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u' \notin V(C)$ and $u' \nsim \{v_5, v_{g-3}\}$ by the choice of (C, P, v_1), $u' \notin \{u, w, w'\}$ as $og(G) \geq 7$, $u' \neq w_1$ as $w_1 \nsim u_2$, and $u' \sim \{u, v_3\}$ to avoid $(u_2u', u_2u_3, u_2u_1u, u_2v_2v_3)$. If $u' \sim u$ then $u' \nsim v_3$ by the choice of (C, P, v_1), and $u' \sim w$ to avoid $(uu', uu_1, vv_gw, uv_{g-2}v_{g-3})$; so $(u_2u_3, u_2u_1, u_2u'w, u_2v_2v_3)$ is a fork, a contradiction. Hence $u' \nsim u$ and $u' \sim v_3$. If $u' \nsim w$ then $(v_3v_2, v_3u', v_3w_1w, v_3v_4v_5)$ would be a fork. So $u' \sim w$. Now $u' \nsim v_{g-1}$ by the choice of (C, P, v_1), and $u' \sim v_1$ to avoid $(v_gu, v_gv_{g-1}, v_gwu', v_gv_1v_2)$. So $(u'w, u'v_1, u'u_2u_3, u'v_3v_4)$ is a fork, a contradiction.

Subcase 2.2. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \in V_3$ and $u_1 \sim w$.

If $w \sim u_3$ then $u_3 \sim v_{g-1}$ to avoid $(wu_3, wu_1, ww_1v_3, wv_gv_{g-1})$, and $v_{g-1} \in V_3$ and $n \geq 4$ by the choice of (C, P, v_1). So $(u_3u_4, u_3v_{g-1}, u_3ww_1, u_3u_2v_2)$ is a fork, a contradiction.

Hence $w \nsim u_3$. Let $x \in N(u_3) \setminus N(u_1)$ such that $x = u_4$ when $n \geq 4$. Then by the choice of (C, P, v_1), $x \notin V(C)$, $x \neq w_1$ and $x \nsim v_g$. So $x \nsim w_1$ to avoid $(v_1u_1, v_1v_2, v_1w_1x, v_1v_gv_{g-1})$. Note that $u_2 \nsim v_4$; for otherwise $v_4 \in V_3$ by the choice of (C, P, v_1), and hence $(u_2v_4, u_2v_2, u_2u_3x, u_2u_1w)$ would be a fork. So $u_2 \nsim C - v_2$ by the choice of (C, P, v_1). By Corollary 4.3, $u_2 \notin V_3$, and let $u \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Since $N(u_3) \not\subseteq N(u)$, we further choose x so that $x \nsim u$ if $x \neq u_4$. Note that $u \sim \{x, v_3\}$ to avoid $(u_2u, u_2u_1, u_2u_3x, u_2v_2v_3)$, $u \sim \{x, w\}$ to avoid $(u_2u, u_2v_2, u_2u_3x, u_2u_1w)$, and $u \sim \{w, v_3\}$ to avoid $(u_2u, u_2u_3, u_2u_1w, u_2v_2v_3)$.

We claim that $u \nsim x$. For, otherwise, $x = u_4$ by the choice of x. So $u \nsim v_3$ by the choice of (C, P, v_1). Hence $u \sim w$, and $u \sim v_{g-1}$ to avoid $(wu, wu_1, ww_1v_3, wv_gv_{g-1})$. Thus, $v_{g-1} \in V_3$ by the choice of (C, P, v_1), and $x \sim v_{g-2}$ to avoid $(ux, uw, uu_2v_2, uv_{g-1}v_{g-2})$. By the choice of (C, P, v_1) again, $n \geq 5$. Hence, $(uv_{g-1}, uw, uu_2v_2, uxu_5)$ is a fork in G, a contradiction.

Hence, $u \sim v_3$ and $u \sim w$; so $u \nsim v_1$ to avoid $(uv_1, uw, uu_2u_3, uv_3v_4)$. Note that $w' \sim v_g$ to avoid $(v_1u_1, v_1v_2, v_1w_1w', v_1v_gv_{g-1})$, $w' \nsim v_4$ and $u \nsim v_5$ by the choice of (C, P, v_1), and $u \sim w'$ to avoid $(v_3u, v_3v_2, v_3v_4v_5, v_3w_1w')$. So $\{w, w'\} \nsim v_{g-2}$ by the choice of (C, P, v_1), and $(v_gw, v_gw', v_gv_1v_2, v_gv_{g-1}v_{g-2})$ is a fork, a contradiction.

Subcase 2.3. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \in V_3$ and $u_1 \nsim w$.

Then we may assume $u_1 \sim w'$. Now $w' \nsim v_g$; otherwise we are back in Subcase 2.2.

We claim that $u_2 \nsim v_4$. For, suppose $u_2 \sim v_4$. Then $v_4 \in V_3$ by the choice of (C, P, v_1), and $u_3 \sim \{w', v_5\}$ to avoid $(u_2u_3, u_2v_2, u_2u_1w', u_2v_4v_5)$. If $u_3 \nsim w'$ then $u_3 \sim v_5$; so $v_5 \in V_3$ and $n \geq 4$ (by the choice of (C, P, v_1)), and hence $(u_2v_2, u_2v_4, u_2u_1w', u_2u_3u_4)$ would be a fork. Hence, $u_3 \sim w'$. Then $u_3 \sim w$ to avoid $(w_1w, w_1v_1, w_1v_3v_4, w_1w'u_3)$, and $u_3 \nsim v_5$ by the choice of (C, P, v_1). Now $(u_2u_1, u_2v_2, u_2u_3w, u_2v_4v_5)$ is a fork, a contradiction.

Thus, by the choice of (C, P, v_1), $u_2 \nsim C \cup (P - \{u_1, u_3\})$. By Corollary 4.3, let $u \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u \notin \{w, w'\}$ to avoid C_5. Let $x \in N(u_3) \setminus \{u_2\}$ such that $x = u_4$ if $n \geq 4$ and, subject to this, $x \nsim u$ when possible. Then $x \notin V(C)$, and $x \neq w_1$ to avoid C_5.

Suppose $u \sim x$ for all choices of x. Then $x = u_4$ (since $N(u_3) \not\subseteq N(u)$), u and u_3 are symmetric, and $\{u, u_3\} \nsim \{v_1, v_3, v_g\}$ by the choice of (C, P, v_1). Now $w' \sim \{u, u_3\}$ to avoid $(u_2u, u_2u_3, u_2v_2v_3, u_2u_1w')$, and assume $w' \sim u$ by symmetry. Then $u \sim w$ or $w' \sim v_4$ to avoid $(w_1w, w_1v_1, w_1w'u, w_1v_3v_4)$. If $u \sim w$ then $v_g \in V_3$ by the choice of (C, P, v_1), and $(ux, uw', uu_2v_2, uwv_g)$ would be a fork. Hence $u \nsim w$, and $w' \sim v_4$. Then $v_4 \in V_3$ by the choice of (C, P, v_1), so $(w'v_4, w'u_1, w'ux, w'w_1w)$ is a fork, a contradiction.

So $u \nsim x$. Then $u \sim v_3$ to avoid $(u_2u, u_2u_1, u_2u_3x, u_2v_2v_3)$, $u \nsim v_1$ to avoid $N(v_2) \subseteq N(u)$, and $u \sim \{w, v_5\}$ to avoid $(v_3u, v_3v_2, v_3w_1w, v_3v_4v_5)$. If $u \sim v_5$ then $u_3 \nsim v_5$ by the choice of (C, P, v_1), so $(u_2u_1, u_2v_2, u_2u_3x, u_2uv_5)$ would be a fork. Thus, $u \nsim v_5$, and $u \sim w$ (so $x \neq w$). Then $u_3 \sim w$ to avoid $(u_2u_1, u_2v_2, u_2u_3x, u_2uw)$; so $v_g \in V_3$ by the choice of (C, P, v_1). Now $w_1 \sim x$ or $u_3 \sim v_{g-1}$ to avoid $(wu, ww_1, wu_3x, wv_gv_{g-1})$. If $w_1 \sim x$ then $(v_1u_1, v_1v_2, v_1w_1x, v_1v_gv_{g-1})$ would be a fork. So $w_1 \nsim x$, and $u_3 \sim v_{g-1}$. Now $v_{g-1} \in V_3$ by the choice of (C, P, v_1); so $(u_3x, u_3v_{g-1}, u_3u_2v_2, u_3ww_1)$ is a fork, a contradiction.

Subcase 2.4. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \notin V_3$.

Suppose $u_1 \nsim w$. Let $w', w'' \in N(u_1) \cap N(w_1)$. Then $v_g \nsim \{w', w''\}$, to avoid $(v_gw, v_gw', v_gv_1v_2, v_gv_{g-1}v_{g-2})$ and $(v_gw, v_gw'', v_gv_1v_2, v_gv_{g-1}v_{g-2})$. So $v_4 \sim \{w', w''\}$ to avoid $(w_1w', w_1w'', w_1v_1v_g, w_1v_3v_4)$, and we may let $v_4 \sim w''$. Then $w' \nsim v_4$ to avoid $(v_4w', v_4w'', v_4v_3v_2, v_4v_5v_6)$, and $u_3 \sim \{w', w''\}$ to avoid $(u_1w', u_1v_1, u_1w''v_4, u_1u_2u_3)$. If $u_3 \nsim w''$ then $u_3 \sim w'$; so $(w_1w'', w_1v_3, w_1w'u_3, w_1v_1v_g)$ would be a fork. Thus, $u_3 \sim w''$. Then $v_4 \in V_3$ by the choice of (C, P, v_1), and $u_3 \sim w'$ to avoid $(w_1w', w_1v_3, w_1v_1v_g, w_1w''u_3)$. Hence, $u_3 \sim w$ to avoid $(w_1w, w_1v_1, w_1w'u_3, w_1v_3v_4)$. Now $(u_3w, w_3w', u_3u_2v_2, u_3w''v_4)$ is a fork, a contradiction.

Hence, $u_1 \sim w$. Then $u_3 \nsim w$; otherwise, $u_3 \sim v_{g-1}$ to avoid $(wu_3, wu_1, wv_gv_{g-1}, ww_1v_3)$, and $\{v_{g-1}, v_g\} \subseteq V_3$ and $n \geq 4$ by the choice of (C, P, v_1). Then $(wv_g, wu_1, wu_3u_4, ww_1v_3)$ is a fork, a contradiction.

We claim that $u_3 \nsim w'$ or $v_4 \nsim w'$. For, suppose $u_3 \sim w'$ and $v_4 \sim w'$. Then $v_4 \in V_3$ by the choice of (C, P, v_1). Let $u \in N(u_3) \backslash N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$, and $u_3 \sim v_5$ or $u \sim w_1$ to avoid $(w'w_1, w'u_1, w'v_4v_5, w'u_3u)$. If $u \sim w_1$ then $(w_1w, w_1v_1, w_1uu_3, w_1v_3v_4)$ would be a fork. So $u \nsim w_1$ and $u_3 \sim v_5$. Then $v_5 \in V_3$ by the choice of (C, P, v_1), and $(u_3u, u_3v_5, u_3w'w_1, u_3u_2v_2)$ is a fork, a contradiction.

In fact, $w' \nsim u_3$; for otherwise, $v_4 \nsim w'$, and so $(w_1w, w_1v_1, w_1v_3v_4, w_1w'u_3)$ would be a fork. Let $w'' \in N(w') \backslash N(w)$. Then $w'' \neq u_3$, $w'' \neq v_g$ to avoid $(v_gw, v_gw', v_gv_1v_2, v_gv_{g-1}v_{g-2})$, and $w'' \neq v_4$ to avoid $(u_1v_1, u_1w, u_1u_2u_3, u_1w'v_4)$. So $w'' \notin V(C)$ by the choice of (C, P, v_1).

If $w'' \sim u_2$ then $w'' \sim v_3$ to avoid $(u_2w'', u_2u_3, u_2u_1w, u_2v_2v_3)$, and $w'' \sim v_5$ to avoid $(v_3w'', v_3v_2, v_3w_1w, v_3v_4v_5)$; so $(u_2u_3, u_2v_2, u_2u_1w, u_2w''v_5)$ is a fork in G, a contradiction. Thus, $w'' \nsim u_2$. So $w'' \sim v_1$ to avoid $(u_1w, u_1v_1, u_1w'w'', u_1u_2u_3)$, $w'' \sim v_3$ to avoid $(v_1v_g, v_1w'', v_1w_1v_3, v_1u_1u_2)$, and $w'' \nsim v_{g-1}$ by the choice of (C, P, v_1). Now $(v_1w'', v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$ is a fork, a contradiction. ∎

We now turn to the case when $u_1 \sim v_{g-1}$ and $w_1 \sim v_3$.

Lemma 5.4. *Suppose $n \geq 3$, $u_1 \sim v_{g-1}$, and $w_1 \sim v_3$. Then for any $v \in N(v_2) \cap N(v_g)$, $|N(v) \cap \{u_1, w_1\}| \neq 1$.*

Proof. By Lemmas 5.2 and 5.3, $\{v_2, v_g\} \nsim P - v_1$. First, suppose there exists $v \in N(v_2) \cap N(v_g)$ such that $v \sim u_1$ and $v \nsim w_1$. Then $v_g \notin V_3$ to avoid $N(v_g) \subseteq N(u_1)$. Hence, $v \nsim u_3$ by the choice of (C, P, v_1), $u_2 \sim v_{g-2}$ to avoid $(u_1v, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$, and $u_2 \nsim w_1$ by the choice of (C, P, v_1). So $(u_1v, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$ is a fork in G, a contradiction.

Now suppose there exists $v \in N(v_2) \cap N(v_g)$ such that $v \nsim u_1$ and $v \sim w_1$. Then $v_2, w_1 \notin V_3$ to avoid $N(v_2) \subseteq N(w_1)$ or $N(w_1) \subseteq N(v_2)$. Let $w \in N(w_1) \setminus N(v_2)$. Then $w \notin V(C)$ by the choice of (C, P, v_1), and $w \notin \{u_1, u_3\}$ since $og(G) \geq 7$. Moreover, $w_1 \nsim u_2$ (so $w \neq u_2$); otherwise, with $v_1 w_1 u_2 \dots u_n$ replacing P, we obtain the situation in the first paragraph.

Note that $w \sim \{u_1, v_g\}$ to avoid $(v_1v_2, v_1v_g, v_1u_1u_2, v_1w_1w)$. Suppose $w \sim u_1$. Then $u_2 \sim v_{g-2}$ to avoid $(u_1u_2, u_1w, u_1v_1v_2, u_1v_{g-1}v_{g-2})$, $w \sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$, $v_4 \nsim w$ by the choice of (C, P, v_1), and $u_3 \sim v$ to avoid $(w_1v, w_1v_1, w_1v_3v_4, w_1wu_3)$. Thus by the choice of (C, P, v_1), $v_2 \in V_3$, a contradiction.

Hence, $w \nsim u_1$, and $w \sim v_g$. Let $v' \in N(v_2) \setminus N(w_1)$. Then $v' \sim \{u_1, v_4\}$ to avoid $(v_2v', v_2v, v_2v_3v_4, v_2v_1u_1)$, and $v' \sim \{u_1, v_g\}$ to avoid $(v_1v_g, v_1w_1, v_1u_1u_2, v_1v_2v')$. Therefore, $v' \sim u_1$ since $og(G) \geq 7$. So $(u_1v', u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$ is a fork, a contradiction. ∎

Lemma 5.5. *Suppose $n \geq 3$, $u_1 \sim v_{g-1}$, and $w_1 \sim v_3$. Then for any $x \in N(v_g) \setminus N(u_1)$ and $y \in N(v_2) \setminus N(w_1)$, $x \sim v_2$ or $y \sim v_g$.*

Proof. For, let $x \in N(v_g) \setminus N(u_1)$ and $y \in N(v_2) \setminus N(w_1)$ such that $x \nsim v_2$ and $y \nsim v_g$. By the choice of (C, P, v_1), $u_2 \nsim v_4$ and $x, y \notin V(C \cup P)$ (also by Lemmas 5.2 and 5.3). Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$.

Case 1. $w_1 \nsim u_2$.

Then $y \sim u_1$ to avoid $(v_1w_1, v_1v_g, v_1u_1u_2, v_1v_2y)$, $y \sim u_3$ to avoid $(u_1y, u_1v_{g-1}, u_1v_1w_1, u_1u_2u_3)$, and $u_2 \sim v_{g-2}$ to avoid $(u_1u_2, u_1y, u_1v_1w_1, u_1v_{g-1}v_{g-2})$. Hence by the choice of (C, P, v_1), $v_2, v_{g-2} \in V_3$ and $v_4 \nsim \{x, y\}$. So $x \sim w_1$ to avoid $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gx)$.

Suppose there exists $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \notin V(C \cup P)$ by the choice of (C, P, v_1), and $v \notin \{x, y\}$ since $og(G) \geq 7$. Now $v \sim \{y, v_5\}$ to avoid $(v_3v, v_3w_1, v_3v_2y, v_3v_4v_5)$. If $v \sim v_5$ then $v \nsim \{x, y\}$ by the choice of (C, P, v_1); so $(v_3v, v_3v_4, v_3v_2y, v_3w_1x)$ would be a fork. Hence $v \nsim v_5$ and $v \sim y$. Then, $v \sim v_1$ to avoid $(u_1v_{g-1}, u_1u_2, u_1v_1w_1, u_1yv)$, $v \sim x$ to avoid $(v_1v, v_1v_2, v_1u_1u_2, v_1v_gx)$, and $u_3 \sim x$ to avoid $(vv_1, vx, vv_3v_4, vyu_3)$. Thus, $v_g \in V_3$ by the choice of (C, P, v_1), and $(u_3u, u_3u_2, u_3xv_g, u_3yv_2)$ is a fork, a contradiction.

So $v_3 \in V_3$ and $w_1 \notin V_3$ (by Corollary 4.3). Let $w \in N(w_1) \setminus \{v_1, v_3, x\}$. Then $w \notin V(C) \cup V(P - \{u_1, u_2\})$ by the choice of (C, P, v_1), and $w \notin \{u_1, u_2, y\}$ as $w_1 \nsim \{u_1, u_2, y\}$. Note that $w \sim \{u_1, v_4\}$ to avoid $(w_1w, w_1x, w_1v_1u_1, w_1v_3v_4)$. If $w \sim u_1$ then $w \sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$; so $(u_3u, u_3w, u_3yv_2, u_3u_2v_{g-2})$ would be a fork. Thus $w \nsim u_1$ and $w \sim v_4$. Then $w \nsim v_g$ by the choice of (C, P, v_1); so $(v_1v_g, v_1v_2, v_1u_1u_2, v_1w_1w)$ is a fork, a contradiction.

Case 2. $w_1 \sim u_2$.

Note that $u_1 \sim y$ or $w_1 \sim x$ to avoid $(v_1u_1, v_1w_1, v_1v_gx, v_1v_2y)$. So by symmetry, assume $u_1 \sim y$. By the choice of (C, P, v_1), $u_2 \nsim v_{g-2}$. So $y \sim u_3$ to avoid $(u_1y, u_1v_1, u_1v_{g-1}v_{g-2}, u_1u_2u_3)$. Hence, $v_2 \in V_3$ by the choice of (C, P, v_1).

Suppose there exists $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \notin V(C \cup P)$ by the choice of (C, P, v_1), and $v \notin \{x, y\}$ since $og(G) \geq 7$. If $v \sim v_5$ then $v \nsim \{y, u_2\}$ by the choice of (C, P, v_1); so $(v_3v, v_3v_4, v_3v_2y, v_3w_1u_2)$ would be a fork. Thus $v \nsim v_5$. Then $v \sim y$ to avoid $(v_3v, v_3w_1, v_3v_4v_5, v_3v_2y)$, $v \sim u_2$ to avoid $(v_3v, v_3v_2, v_3v_4v_5, v_3w_1u_2)$, and $v \sim u$ to avoid $(yv, yv_2, yu_1v_{g-1}, yu_3u)$. So let $u_3' \in N(u_3) \setminus N(v)$. By the choice of (C, P, v_1), $n = 3$ and $u_3' \notin$

$V(C\cup P)$. Now $u_3'\sim u_1$ to avoid $(yv, yv_2, yu_1v_{g-1}, yu_3'u_3)$, and $v\nsim v_1$ to avoid $N(v_2)\subseteq N(v)$; so $(u_1u_3', u_1v_{g-1}, u_1u_2v, u_1v_1v_2)$ is a fork, a contradiction.

Thus, $v_3\in V_3$ and $w_1\notin V_3$ (by Corollary 4.3). Let $w\in N(w_1)\setminus\{u_2,v_1,v_3\}$. Then $w\notin V(C)\cup V(P-u_1)$ by the choice of (C,P,v_1), $w\neq u_1$ as $og(G)\geq 7$, and $w\neq y$ as $w_1\nsim y$. Note that $w_1\nsim x$ (so $u\neq x$); otherwise $x\sim u_3$ to avoid $(w_1x, w_1v_1, w_1u_2u_3, w_1v_3v_4)$, and so $(u_3u, u_3u_2, u_3xv_g, u_3yv_2)$ would be a fork. Also note that $w\sim\{v_4,v_g\}$ to avoid $(w_1w, w_1u_2, w_1v_1v_g, w_1v_3v_4)$. If $w\sim v_4$ then $w\nsim\{u_1,v_g\}$ by the choice of (C,P,v_1); so $(v_1u_1, v_1v_2, v_1w_1w, v_1v_gx)$ is a fork, a contradiction. Thus, $w\nsim v_4$, and $w\sim v_g$. So $w\nsim u_3$ by the choice of (C,P,v_1), and $(w_1w, w_1v_1, w_1u_2u_3, w_1v_3v_4)$ is a fork, a contradiction. ∎

Lemma 5.6. *Suppose $n\geq 3$, $w_1\sim v_3$, and $u_1\sim v_{g-1}$. Then $(N(v_2)\setminus N(w_1))\cap(N(v_g)\setminus N(u_1))=\emptyset$.*

Proof. For, let $v\in(N(v_2)\setminus N(w_1))\cap(N(v_g)\setminus N(u_1))$. Then $v\neq u_2$ by Lemma 5.2, and $v\nsim u_2$ since $og(G)\geq 7$. Thus, since $v_1\notin V_3$, $v\notin V_3$ by the choice of (C,P,v_1). Let $v',v''\in N(v)\setminus\{v_2,v_g\}$ be distinct. Then $v',v''\notin V(C)$ by the choice of (C,P,v_1), $\{v',v''\}\sim\{v_3,v_{g-1}\}$ to avoid $(vv', vv'', vv_2v_3, vv_gv_{g-1})$. By symmetry, let $v'\sim\{v_3,v_{g-1}\}$. Then $v'\notin V(P)\setminus\{u_1,u_2,v_1\}$ by the choice of (C,P,v_1), $v',v''\notin\{u_1,w_1\}$ as $v\nsim\{u_1,w_1\}$, and $v',v''\notin\{u_2,v_1\}$ and $\{v',v''\}\nsim\{u_1,w_1\}$ as $og(G)\geq 7$.

We claim that $w_1\nsim u_2$. For, suppose $w_1\sim u_2$. By symmetry let $v'\sim v_3$. Then $u_2\sim v'$ to avoid $(v_3v', v_3v_2, v_3v_4v_5, v_3w_1u_2)$, and $v\sim u_3$ to avoid $(u_2u_3, u_2w_1, u_2v'v, u_2u_1v_{g-1})$. So $\{v_2,v_g\}\subseteq V_3$ by the choice of (C,P,v_1). Let $u\in N(u_3)\setminus N(v')$ such that $u=u_4$ if $n\geq 4$. Thus $u\notin C$, and so $(vv_2, vv', vu_3u, vv_gv_{g-1})$ is a fork in G, a contradiction.

Let $w\in N(w_1)\setminus N(v_2)$. Then $w\notin V(C)\cup V(P-\{u_1,u_2\})$ by the choice of (C,P,v_1), $w\notin\{u_2,v\}$ as $w_1\nsim\{u_2,v\}$, and $w\notin\{u_1,v',v''\}$ since $og(G)\geq 7$. Note that $w\sim\{u_1,v_g\}$ to avoid $(v_1v_2, v_1v_g, v_1u_1u_2, v_1w_1w)$. If $w\sim u_1$ then $w\sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$; hence, $w\nsim v_g$ by the choice of (C,P,v_1), and replacing P with $v_1w_1wu_3\dots u_n$, we get a contradiction to the above claim that $w_1\nsim u_2$ (because $w\sim u_1$). Thus, $w\nsim u_1$ and $w\sim v_g$. Hence, by the choice of (C,P,v_1), $w\nsim v_3$ and $u_1\notin V_3$. Let $u\in N(u_1)\setminus\{u_2,v_1,v_{g-1}\}$. Then $u\notin V(C\cup P)\cup\{v,w,w_1\}$, by the choice of (C,P,v_1) and the fact that $u_1\nsim\{v,w\}$.

Case 1. $u\nsim u_3$.

Then $u\sim w_1$ and $u\sim v_2$ to avoid $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$ and $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1v_2)$, respectively. So $u\nsim v_{g-2}$ by the choice of (C,P,v_1), and $u_2\sim v_{g-2}$ to avoid $(u_1u, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$. Hence $v_{g-2}\in V_3$ by the choice of (C,P,v_1). If $u_2\in V_3$ then $v_{g-1}\in V_3$ by the choice of (C,P,v_1), contradicting Corollary 4.3. Hence $u_2\notin V_3$.

Let $u'\in N(u_2)\setminus\{u_1,u_3,v_{g-2}\}$. Then $u'\notin V(C\cup P)$ by the choice of (C,P,v_1), $u'\notin\{u,v,w\}$ since $og(G)\geq 7$, and $u'\neq w_1$ as $u_2\nsim w_1$. Let $z\in N(u_3)\setminus N(u_1)$ such that $z\notin V(C)$. (Note that such z does exist as otherwise $n\geq 4$ and we may choose $z=u_4$.) Then $z\notin\{u,v,w,w_1\}$ by the choice of (C,P,v_1), and $z\neq u'$ as $og(G)\geq 7$.

Suppose $u'\nsim z$. Then $u'\sim v_1$ to avoid $(u_2u', u_2v_{g-2}, u_2u_3z, u_2u_1v_1)$, $u'\sim v_{g-1}$ to avoid $(v_1v_2, v_1w_1, v_1u'u_2, v_1v_gv_{g-1})$, and $u'\sim\{v_{g-3},w\}$ to avoid $(v_{g-1}u', v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gw)$. If $u'\sim w$ then $(u'v_{g-1}, u'w, u'v_1v_2, u'u_2u_3)$ would be a fork. So $u'\sim v_{g-3}$, and $(u'v_{g-1}, u'v_{g-3}, u'v_1v_2, u'u_2u_3)$ is a fork, a contradiction.

Thus, $u' \sim z$ for all $z \in N(u_3) \setminus N(u_1)$ such that $z \notin V(C)$. Let $u'_3 \in N(u_3) \setminus N(u')$. Then by the choice of z, $u'_3 \sim u_1$ or $u'_3 = v_{g-3}$. If $u'_3 \sim u_1$ then $u'_3 \notin V(C)$, and $(u_1u, u_1v_1, u_1u'_3u_3, u_1v_{g-1}v_{g-2})$ would be a fork. So $u'_3 \nsim u_1$ and $u'_3 = v_{g-3}$, and thus we may assume $z = u_4$.

Now u' is symmetric to u_3. Applying the argument above for u_3 to u', $u'' = v_{g-3}$ for any $u'' \in N(u') \setminus N(u_1)$, a contradiction as $v_{g-3} \in V_3$ by the choice of (C, P, v_1).

Case 2. $u \sim u_3$.

Then $u \nsim \{v_2, v_g\}$ by the choice of (C, P, v_1). So u and u_2 are symmetric, and $v_{g-2} \sim \{u, u_2\}$ to avoid $(u_1u, u_1u_2, u_1v_1v_2, u_1v_{g-1}v_{g-2})$. By symmetry, assume $u \sim v_{g-2}$. Then by the choice of (C, P, v_1), $v_{g-2} \in V_3$, $u \nsim w_1$, and $u \nsim C - \{v_{g-2}, v_{g-3}\}$. Since $og(G) \geq 7$, $u \nsim \{u_2, v, v_{g-3}, w\}$.

If $u \in V_3$ then $v_{g-1} \in V_3$ by the choice of (C, P, v_1), contradicting Corollary 4.3. Hence, $u \notin V_3$. Let $u' \in N(u) \setminus \{u_1, u_3, v_{g-2}\}$. Then $u' \notin V(C \cup P) \cup \{u_2, v, w, w_1\}$ by the assumption $og(G) \geq 7$ and the choice of (C, P, v_1). Let $z \in N(u_3) \setminus N(u_1)$ such that $z = u_4$ if $n \geq 4$. Then $z \notin V(C)$, and $z \notin \{v_1, v, w_1\}$ and $z \nsim \{v_1, v_2, v_3, v_{g-1}, v_g\}$ by the choice of (C, P, v_1). So $u' \sim \{z, v_1\}$ to avoid $(uu', uv_{g-2}, uu_3z, uu_1v_1)$.

If $u' \sim v_1$ then $u' \nsim v_{g-3}$ by the choice of (C, P, v_1), $u' \sim v_{g-1}$ to avoid $(v_1w_1, v_1v_2, v_1u'u, v_1v_gv_{g-1})$, and $v \sim u'$ to avoid $(v_{g-1}u', v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gv)$; so $(u'v_{g-1}, u'v, u'v_1w_1, u'uu_3)$ is a fork, a contradiction.

Hence, $u' \nsim v_1$ and $u' \sim z$. Let $u'_2 \in N(u_2) \setminus N(u)$. Then $u'_2 \notin V(C \cup P)$ by the choice of (C, P, v_1), $u'_2 \neq z$ since $og(G) \geq 7$, and $u'_2 \notin \{v, w_1\}$ since $u_2 \nsim \{v, w_1\}$. Now $u'_2 \sim \{v_1, v_{g-1}\}$ to avoid $(u_1u, u_1v_{g-1}, u_1v_1v_2, u_1u_2u'_2)$. If $u'_2 \sim v_1$ then $u'_2 \sim v_3$ to avoid $(v_1u'_2, v_1v_g, v_1u_1u, v_1v_2v_3)$; so $(v_3w_1, v_3v_2, v_3u'_2u_2, v_3v_4v_5)$ would be a fork. So $u'_2 \nsim v_1$ and $u'_2 \sim v_{g-1}$. Then $u'_2 \nsim v_{g-3}$; for otherwise replacing P with $v_{g-1}u_1uu_3 \dots u_n$, we get a contradiction to Lemma 5.3. Hence $u'_2 \sim w$ to avoid $(v_{g-1}u'_2, v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gw)$, and $u' \sim \{u_2, v_{g-1}\}$ to avoid $(u_1u_2, u_1v_{g-1}, u_1v_1v_2, u_1uu')$. If $u' \sim u_2$ then $u' \sim w$ to avoid $(u_2u', u_2u_3, u_2u_1v_1, u_2u'_2w)$; so $(u'z, u'u_2, u'uv_{g-2}, u'wv_g)$ would be a fork. Thus, $u' \nsim u_2$ and $u' \sim v_{g-1}$. Then $u' \sim v_{g-3}$ to avoid $(v_{g-1}u', v_{g-1}v_g, v_{g-1}u_1u_2, v_{g-1}v_{g-2}v_{g-3})$. Now, replacing P with $v_{g-1}u_1uu_3 \dots u_n$, we get a contradiction to Lemma 5.3. ∎

6 Final reduction

We now show that $n \leq 2$. First, we need the following lemma.

Lemma 6.1. *Suppose $n \geq 3$. If $w_1 \sim \{v_3, v_{g-1}\}$ and $u_1 \nsim \{v_3, v_{g-1}\}$. Then $w_1 \nsim u_2$.*

Proof. By symmetry assume $w_1 \sim v_3$. Suppose $w_1 \sim u_2$.

Case 1. $w_1 \notin V_3$.

Let $w \in N(w_1) \setminus \{u_2, v_1, v_3\}$. Then $w \notin V(C \cup P)$ by the choice of (C, P, v_1) and the fact that $og(G) \geq 7$. Note that the exceptional case of Lemma 5.3 does not occur for C even if we change P to another path P' with $V(P' \cap C) = \{v_1\}$. For, otherwise, $v_2 \in V_3$ and there exists $w'_1 \in V_3$ such that $w'_1 \sim v_1$ and $w'_1 \in v_3$. Then $\{v_2, w'_1\} \nsim \{u_2, w\}$ to avoid $N(v_2) \subseteq N(w_1)$ or

$N(w_1') \subseteq N(w_1)$. So $u_2 \sim v_4$ to avoid $(v_3v_2, v_3w_1', v_3v_4v_5, v_3w_1u_2)$, and hence $v_4 \in V_3$ by the choice of (C, P, v_1). Thus, $(v_3v_2, v_3w_1', v_3v_4v_5, v_3w_1w)$ is a fork, a contradiction.

We claim that $w \sim u_3$. For, suppose $w \nsim u_3$. Then $\{w, u_2\} \sim v_4$ to avoid $(w_1w, w_1v_1, w_1u_2u_3, w_1v_3v_4)$. If $w \sim v_4$ then $w \nsim v_g$ by the choice of (C, P, v_1), $u_2 \sim v_g$ to avoid $(w_1w, w_1v_3, w_1u_2u_3, w_1v_1v_g)$, $v_g \in V_3$ by the choice of (C, P, v_1), and $u_3 \sim v_{g-1}$ to avoid $(u_2u_3, u_2u_1, u_2w_1v_3, u_2v_gv_{g-1})$; so $n \geq 4$ (as $d(u_n, C) \geq 2$), and $(u_2v_g, u_2u_1, u_2u_3u_4, u_2w_1v_3)$ is a fork in G, a contradiction. Thus, $w \nsim v_4$ and $u_2 \sim v_4$. Hence $v_4 \in V_3$ by the choice of (C, P, v_1). If $u_3 \nsim v_5$ then let $u_3' \in N(u_3) \setminus N(u_1)$ such that $u_3' = u_4$ if $n \geq 4$; now $u_3' \notin V(C)$, and $u_3' \sim w_1$ to avoid $(u_2u_1, u_2w_1, u_2v_4v_5, u_2u_3u_3')$, which implies the fork $(w_1w, w_1v_1, w_1v_3v_4, w_1u_3'u_3)$, a contradiction. Hence $u_3 \sim v_5$, $v_5 \in V_3$ by the choice of (C, P, v_1), and $n \geq 4$ since $d(u_n, C) \geq 2$. Now $w \sim u_1$ to avoid $(u_2u_1, u_2v_4, u_2u_3u_4, u_2w_1w)$. Let $u \in N(u_1) \setminus N(w_1)$. Note that $u \notin V(C \cup P)$ by the choice of (C, P, v_1) (and since $u_1 \nsim \{v_3, v_{g-1}\}$). So $u \sim u_3$ to avoid $(u_2w_1, u_2v_4, u_2u_3u_4, u_2u_1u)$, and $u_4 \sim v_6$ to avoid $(u_3u_4, u_3u, u_3u_2w_1, u_3v_5v_6)$. Hence $v_6 \in V_3$ by the choice of (C, P, v_1) and $n \geq 5$ since $d(u_n, C) \geq 2$; so $(u_3u, u_3v_5, u_3u_4u_5, u_3u_2w_1)$ is a fork in G, a contradiction.

Then $w \nsim v_2$; for otherwise, $v_2 \in V_3$ by the choice of (C, P, v_1) which would imply $N(v_2) \subseteq N(w_1)$. Also $u_2 \nsim v_g$. For if $u_2 \sim v_g$ then $v_g \in V_3$ by the choice of (C, P, v_1), $v_{g-1} \sim u_3$ to avoid $(u_2u_3, u_2u_1, u_2w_1v_3, u_2v_gv_{g-1})$, and $n \geq 4$ since $d(u_n, C) \geq 2$; so $(u_2u_1, u_2v_g, u_2u_3u_4, u_2w_1v_3)$ is a fork in G, a contradiction.

We claim that $u_2 \nsim v_4$. For, suppose $u_2 \sim v_4$. Then $v_4 \in V_3$ by the choice of (C, P, v_1). Let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$, and $u \neq u_1$ since $og(G) \geq 7$. If $u \sim u_1$ then $u \nsim v_2$ (otherwise with $v_1u_1uu_5 \ldots u_n$ replacing P we get a contradiction to Lemma 5.3), $u \sim v_g$ to avoid $(v_1w_1, v_1v_2, v_1u_1u, v_1v_gv_{g-1})$, $v_g \in V_3$ by the choice of (C, P, v_1), and $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$; so $(u_1u, u_1w, u_1u_2v_4, u_1v_1v_2)$ is a fork, a contradiction. Hence $u \nsim u_1$, $u_3 \sim v_5$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2v_4v_5)$, and $w \nsim v_g$ by the choice of (C, P, v_1) (minimality of C). Then $v_5 \in V_3$ by the choice of (C, P, v_1), and so $u = u_4$. If $w \nsim u_1$ then $u_4 \sim v_6$ to avoid $(u_3u_4, u_3w, u_3u_2u_1, u_3v_5v_6)$ and, hence, $v_6 \in V_3$ by the choice of (C, P, v_1), and $n \geq 5$ as $d(u_n, C) \geq 2$; so $(u_3w, u_3v_5, u_3u_4u_5, u_3u_2u_1)$ is a fork, a contradiction. Thus $w \sim u_1$. Let $w' \in N(w) \setminus N(u_2)$. Then $w' \notin V(C - v_2) \cup V(P)$ by the choice of (C, P, v_1), and $w' \neq v_2$ as $w \nsim v_2$. If $w' \nsim u_4$ then $u_4 \sim v_6$ to avoid $(u_3u_2, u_3u_4, u_3v_5v_6, u_3ww')$; so $n \geq 5$ and $(u_3u_2, u_3v_5, u_3u_4u_5, u_3ww')$ is a fork, a contradiction. Hence, $w' \sim u_4$, and so $w' \nsim \{v_1, v_3\}$ by the choice of (C, P, v_1). Now $(w_1u_2, w_1v_3, w_1ww', w_1v_1v_g)$ is a fork, a contradiction.

Suppose $w \sim v_4$. Then $v_4 \in V_3$, $w \nsim v_g$ and $u_3 \nsim v_{g-1}$ by the choice of (C, P, v_1); so $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$. Suppose $u_3 \nsim v_5$, and let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$, $u_1 \sim u$ to avoid $(wu_1, ww_1, wu_3u, wv_4v_5)$, $u \sim v_2$ to avoid $(u_1u, u_1u_2, u_1wv_4, u_1v_1v_2)$, $u \sim v_g$ to avoid $(u_1u, u_1u_2, u_1wv_4, u_1v_1v_g)$; so $(uu_3, uu_1, uv_2v_3, uv_gv_{g-1})$ is a fork, a contradiction. Hence, $u_3 \sim v_5$, $v_5 \in V_3$ by the choice of (C, P, v_1), and $n \geq 4$ as $d(u_n, C) \geq 2$. Let $u_1' \in N(u_1) \setminus N(w_1)$. Then, since $u_1 \nsim \{v_3, v_{g-1}\}$ and $og(G) \geq 7$, $u_1' \notin V(C \cup P)$ by the choice of (C, P, v_1). Now $u_1' \sim u_3$ to avoid $(wv_4, ww_1, wu_3u_4, wu_1u_1')$; so $(u_3u_1', u_3u_4, u_3u_2w_1, u_3v_5v_4)$ is a fork, a contradiction.

Hence, $w \nsim v_4$, $w \sim v_g$ to avoid $(w_1w, w_1u_2, w_1v_3v_4, w_1v_1v_g)$, and $v_g \in V_3$ by the choice of (C, P, v_1). If $w \sim u_1$ then $u_3 \sim v_{g-1}$ to avoid $(wu_1, wu_3, ww_1v_3, wv_gv_{g-1})$, and hence $n \geq 4$ as $d(u_n, C) \geq 2$; so $(wu_1, wv_g, wu_3u_4, ww_1v_3)$ would be a fork. Thus $w \nsim u_1$. Let $u \in N(u_3) \setminus \{u_2, w\}$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$. By Corollary 4.3, $\{u_2, u_3, w\} \nsubseteq V_3$.

First, suppose $u_3 \notin V_3$, and let $u' \in N(u_3) \setminus \{u_2, u, w\}$. In this case, we will not use the restriction "$u = u_4$ if $n \geq 4$", u and u' are symmetric. Suppose $\{u, u'\} \sim w_1$ and by symmetry assume $u' \sim w_1$. Then $u' \notin V(C)$, $u' \sim v_4$ to avoid $(w_1u', w_1u_2, w_1v_1v_g, w_1v_3v_4)$, and $v_4 \in V_3$ by the choice of (C, P, v_1); so $(u_3u, u_3u_2, u_3wv_g, u_3u'v_4)$ is a fork, a contradiction. Hence, $\{u, u'\} \nsim w_1$ and by symmetry, assume $u_1 \sim u'$. Then $u' \neq u_4$ by the choice of (C, P, v_1), and $u' \nsim v_2$ by Lemma 5.3 (with $v_1u_1u'u_3 \ldots u_n$ replacing P). Since $u_1 \sim \{u, u'\}$ to avoid $(u_3u, u_3u', u_3wv_g, u_3u_2u_1)$, $(v_1v_2, v_1w_1, v_1u_1u', v_1v_gv_{g-1})$ is a fork, a contradiction.

Now suppose $u_3 \in V_3$ and $u_2 \notin V_3$. Then $u \nsim w_1$ to avoid $N(u_3) \subseteq N(w_1)$. Let $u' \in N(u_2) \setminus \{u_1, u_3, w_1\}$. Then, since $u_2 \nsim \{v_2, v_4, v_g\}$ and $og(G) \geq 7$, $u' \notin V(C \cup P)$ by the choice of (C, P, v_1). If $u' \nsim \{v_1, v_3\}$ then $u' \sim w$ to avoid $(w_1w, w_1v_1, w_1v_3v_4, w_1u_2u')$ and $u' \nsim u$ to avoid $N(u_3) \subseteq N(u')$; so $(wu', wv_g, ww_1v_3, wu_3u)$ is a fork, a contradiction. If $u' \nsim v_3$ and $u' \sim v_1$ then, since $w \nsim u_1$, $w \sim u'$ to avoid $(v_1u_1, v_1u', v_1v_2v_3, v_1v_gw)$; so $u' \nsim u$ to avoid $N(u_3) \subseteq N(u')$, and $(wu', wv_g, ww_1v_3, wu_3u)$ is a fork, a contradiction. Thus $u' \sim v_3$, and $u' \nsim v_{g-1}$ by the choice of (C, P, v_1). Now $u' \nsim w$; otherwise $u' \nsim u$ to avoid $N(u_3) \subseteq N(u')$, and $(wu', ww_1, wv_gv_{g-1}, wu_3u)$ would be a fork. Then $u' \sim v_5$ to avoid $(v_3u', v_3v_2, v_3w_1w, v_3v_4v_5)$, and $u \sim \{u', u_1\}$ to avoid $(u_2u_1, u_2w_1, u_2u'v_5, u_2u_3u)$. If $u \sim u_1$ then $u \nsim v_2$ by Lemma 5.3 (with $v_1u_1uu_3 \ldots u_n$ replacing P); so $(v_1v_2, v_1w_1, v_1v_gv_{g-1}, v_1u_1u)$ is a fork, a contradiction. Hence $u \nsim u_1$ and $u \sim u'$. Then $u \nsim v_2$ by Lemma 5.3 (with $v_3u'uu_3 \ldots u_n$ replacing P), and $u \sim v_4$ to avoid $(v_3v_4, v_3v_2, v_3u'u, v_3w_1w)$. So $v_4 \in V_3$ by the choice of (C, P, v_1), which implies $N(v_4) \subseteq N(u')$, contradicting Lemma 3.2.

Thus, $u_2, u_3 \in V_3$ and $w \notin V_3$. Let $w' \in N(w) \setminus \{u_3, v_g, w_1\}$. Then $w' \notin V(C \cup P)$ and $w' \nsim v_5$ by the choices of (C, P, v_1). Now $w' \sim \{v_3, v_{g-1}\}$ to avoid $(ww', wu_3, ww_1v_3, wv_gv_{g-1})$. If $w' \sim v_3$ then $(v_3w', v_3v_2, v_3v_4v_5, v_3w_1u_2)$ would be a fork. Hence $w' \nsim v_3$ and $w' \sim v_{g-1}$, and $w' \nsim v_1$ to avoid $N(v_g) \subseteq N(w')$. Note that $u \nsim w_1$ to avoid $N(u_3) \subseteq N(w_1)$, and $w' \sim u$ to avoid $(ww', wv_g, ww_1v_3, wu_3u)$. So $(w_1u_2, w_1v_1, w_1ww', w_1v_3v_4)$ is a fork in G, a contradiction.

Case 2. $w_1 \in V_3$.

Then $v_2 \in V_3$ by the choice of (C, P, v_1), and $v_3 \notin V_3$ by Corollary 4.3. Let $x \in N(v_2) \setminus \{v_1, v_3\}$ and $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Note that $x \notin V(C \cup P)$ by the choice of (C, P, v_1) and Lemma 5.2, $y \notin V(C \cup P)$ by the choice of (C, P, v_1) and because $v_3 \nsim u_1$, and $w_1 \neq x \neq y$ since $og(G) \geq 7$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$, and $u \neq y$ as $og(G) \geq 7$.

We claim that $u_2 \nsim C$. For, suppose $u_2 \sim C$. Then, since $og(G) \geq 7$, $u_2 \sim \{v_4, v_g\}$ by the choice of (C, P, v_1). If $u_2 \sim v_g$ then $v_g \in V_3$ by the choice of (C, P, v_1); $(u_2u_1, u_2v_g, u_2u_3u, u_2w_1v_3)$ would be a fork. Hence $u_2 \nsim v_g$ and $u_2 \sim v_4$, and thus $v_4 \in V_3$ by the choice of (C, P, v_1). Now $u_3 \sim v_5$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2v_4v_5)$. So $v_5 \in V_3$ by the choice of (C, P, v_1), and $n \geq 4$ as $d(u_n, C) \geq 2$. Let $z \in N(u_4) \setminus \{u_3\}$ such that $z = u_5$ if $n \geq 5$ (z is arbitrary if $n = 4$). Then $z \notin V(C)$, $z \notin \{u_1, u_2, w_1\}$ by the choice of (C, P, v_1), and $z \nsim u_2$ to avoid $(u_2u_1, u_2w_1, u_2zu_4, u_2v_4v_5)$. By Lemma 4.3, let $u' \in N(u_3) \setminus \{u_2, u_4, v_5\}$. Then $u' \notin V(C \cup P)$ by the choice of (C, P, v_1) and because $og(G) \geq 7$. Hence $z \sim u'$ to avoid $(u_3u', u_3v_5, u_3u_2w_1, u_3u_4z)$, and so $z = u_5$ to avoid $N(u_4) \subseteq N(u')$ (as z is arbitrary when $n = 4$). Moreover, $v_6 \sim \{u', u_4\}$ to avoid $(u_3u', u_3u_4, u_3v_5v_6, u_3u_2w_1)$. By the symmetry between u' and u_4, let $u' \sim v_6$. Then $v_6 \in V_3$ by the choice of (C, P, v_1). Let $u'' \in N(u_4) \setminus N(u')$. Then $u'' \sim u_2$ to avoid $(u_3u', u_3v_5, u_3u_2w_1, u_3u_4u'')$. So $(u_2u'', u_2v_4, u_2u_3u', u_2w_1v_1)$ is a fork, a contradiction.

Now we show $y \nsim v_5$. For, suppose $y \sim v_5$. Then $u_2 \nsim y$; for otherwise, $u \sim y$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yv_5)$, and $(yu, yv_5, yu_2u_1, yv_3v_2)$ would be a fork. Hence, $x \sim \{v_4, y\}$ to avoid $(v_3y, v_3v_4, v_3w_1u_2, v_3v_2x)$. If $x \sim v_4$ then let $v \in N(y) \backslash N(v_4)$; now $(v_3v_2, v_3v_4, v_3w_1u_2, v_3yv)$ is a fork, a contradiction. So $x \sim y$. Let $v \in N(v_4) \setminus N(y)$; then $(v_3v_2, v_3y, v_3w_1u_2, v_3v_4v)$ is a fork, a contradiction.

Then $y \nsim u$ to avoid $(v_3v_2, v_3w_1, v_3yu, v_3v_4v_5)$, $y \sim u_2$ to avoid $(v_3y, v_3v_2, v_3v_4v_5, v_3w_1u_2)$, and $x \sim \{y, v_4\}$ to avoid $(v_3y, v_3w_1, v_3v_4v_5, v_3v_2x)$. If $N(y) = \{u_2, v_3, x\}$ for all $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$, then $v_3 \in V_4$, and hence v_4, v_3 contradict Lemma 3.4. Thus, let y be chosen so that there exists $y' \in N(y) \setminus \{u_2, v_3, x\}$. Then $y' \neq v_5$ (since $y \nsim v_5$), $y' \neq v_1$ (since $y \nsim v_1$ to avoid $N(w_1) \subseteq N(y)$), and $y' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3yy')$.

We claim that $x \nsim v_4$ (and hence $x \sim y$). For, otherwise $u_3 \nsim y'$ by the choice of (C, P, v_1), $y' \sim v_6$ to avoid $(v_4y', v_4x, v_4v_3w_1, v_4v_5v_6)$, $y' \nsim u_1$ by the choice of (C, P, v_1), and $u \sim y$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yy')$. So $(yu, yu_2, yv_3v_2, yy'v_6)$ is a fork, a contradiction.

We also claim that $u_3 \nsim x$. For, suppose $u_3 \sim x$. Then $y' \nsim u_3$ to avoid $(u_3y', u_3u, u_3xv_2, u_3u_2w_1)$, $x \nsim v_g$ to avoid $(xv_g, xv_2, xyy', xu_3u)$, $x \sim u_1$ to avoid $(v_1w_1, v_1u_1, v_1v_gv_{g-1}, v_1v_2x)$, and $y' \sim u_1$ to avoid $(xu_1, xv_2, xu_3u, xyy')$. Hence $(u_1u_2, u_1x, u_1v_1v_g, u_1y'v_4)$ is a fork, a contradiction.

Then $x \sim u_1$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yx)$. Moreover, $y' \sim u_3$; for, otherwise, $y' \sim u_1$ to avoid $(u_2u_1, u_2w_1, u_2yy', u_2u_3u)$, and so $(u_1v_1, u_1x, u_1u_2u_3, u_1y'v_4)$ is a fork in G, a contradiction. Hence $v_4 \in V_3$ by the choice of (C, P, v_1).

We now show that $u_1 \in V_3$. For, suppose $u_1 \notin V_3$, and let $u' \in N(u_1) \setminus \{u_2, v_1, x\}$. Then, since $u_1 \nsim \{v_3, v_{g-1}\}$ and $og(G) \geq 7$, $u' \notin V(C \cup P) \cup \{w_1, y\}$ by the choice of (C, P, v_1). Moreover, $u' \neq y'$ to avoid $(y'y, y'v_4, y'u_3u, y'u_1v_1)$. Now $u' \sim \{u_3, y\}$ to avoid $(u_2y, u_2w_1, u_2u_3u, u_2u_1u')$. If $u' \sim u_3$ then $(u_3u', u_3u, u_3u_2w_1, u_3y'v_4)$ would be a fork. So $u' \nsim u_3$ and $u' \sim y$. Now $(v_3v_2, v_3w_1, v_3v_4v_5, v_3yu')$ is a fork, a contradiction.

If $v_1 \in V_4$ then v_g, v_1 contradict Lemma 3.4. So $v_1 \notin V_4$, and let $v \in N(v_1) \setminus \{v_2, v_g, u_1, w_1\}$. Then $v \notin V(C \cup P)$ by the choice of (C, P, v_1), $v \notin \{x, y'\}$ since $og(G) \geq 7$, and $v \neq y$ (as $y \nsim v_1$). Note that $v \sim \{u_2, v_{g-1}\}$ to avoid $(v_1v, v_1v_2, v_1u_1u_2, v_1v_gv_{g-1})$, and $v \sim \{u_2, v_3\}$ to avoid $(v_1v, v_1v_g, v_1u_1u_2, v_1v_2v_3)$. By the choice of (C, P, v_1), $v \nsim v_3$ or $v \nsim v_{g-1}$; so $v \sim u_2$. Then $v \nsim v_3$ (to avoid $N(w_1) \subseteq N(v)$), and $v \sim u$ to avoid $(u_2v, u_2u_1, u_2w_1v_3, u_2u_3u)$. Now $x \nsim v$ to avoid $N(u_1) \subseteq N(v)$; so $(v_1u_1, v_1w_1, v_1vu, v_1v_2x)$ is a fork, a contradiction. ∎

Lemma 6.2. *Suppose $n \geq 3$. If $w_1 \sim \{v_3, v_{g-1}\}$ then $u_1 \sim \{v_3, v_{g-1}\}$.*

Proof. For, suppose $w_1 \sim \{v_3, v_{g-1}\}$ and $u_1 \nsim \{v_3, v_{g-1}\}$. By symmetry assume $w_1 \sim v_3$. Then $w_1 \nsim u_2$ by Lemma 6.1; so $w_1 \nsim P - v_1$ by the choice of (C, P, v_1). Moreover, $v_2 \nsim P - v_1$; otherwise, by Lemma 5.3, $w_1, v_2 \in V_3$, $v_2 \sim u_2$, and $w_1 \nsim u_2$, and hence by replacing C with $v_1w_1, v_3 \ldots v_gv_1$ we get a contradiction to Lemma 6.1 (since $v_2 \sim u_2$).

Therefore, $u_2 \sim v_g$ to avoid $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$. So by the choice of (C, P, v_1), $v_g \in V_3$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin V(C)$, and $u \neq w_1$ since $og(G) \geq 7$. Let $x_1 \in N(w_1) \setminus N(v_2)$, and $x_2 \in N(v_2) \setminus N(w_1)$. Then $x_1, x_2 \notin V(C \cup P)$. Note that $u_1 \sim x_i$ for $i = 1, 2$, to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1x_1)$ and $(v_1u_1, v_1w_1, v_1v_gv_{g-1}, v_1v_2x_2)$. So $u \notin \{x_1, x_2\}$ as $u \notin N(u_1)$.

Claim 1. $N(w_1) \setminus \{x_1\} = N(v_2) \setminus \{x_2\}$.

Suppose there exists $v \in N(v_2)\setminus(\{x_2\} \cup N(w_1))$. Then $u_1 \sim v$ to avoid $(v_1u_1, v_1w_1, v_1v_gv_{g-1}, v_1v_2v)$, and $v_4 \sim \{x_2, v\}$ to avoid $(v_2x_2, v_2v, v_2v_1v_g, v_2v_3v_4)$. By symmetry, let $v_4 \sim v$. Note that $u_3 \nsim \{v, x_2\}$ by the choice of (C, P, v_1). So $v_4 \sim x_2$ to avoid $(u_1v_1, u_1x_2, u_1u_2u_3, u_1vv_4)$. Now $(v_4x_2, v_4v, v_4v_3w_1, v_4v_5v_6)$ is a fork, a contradiction.

Now suppose there exists $v \in N(w_1) \setminus (\{x_1\} \cup N(v_2))$. Then $u_1 \sim v$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1v)$, and $v_4 \sim \{x_1, v\}$ to avoid $(w_1x_1, w_1v, w_1v_1v_g, w_1v_3v_4)$. By symmetry, let $v \sim v_4$. Then $v_4 \nsim x_1$ to avoid $(v_4x_1, v_4v, v_4v_3v_2, v_4v_5v_6)$, and $u_3 \sim \{x_1, v\}$ to avoid $(u_1x_1, u_1v_1, u_1u_2u_3, u_1vv_4)$. If $u_3 \nsim v$ then $u_3 \sim x_1$, and $(w_1v, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ would be a fork. Hence $u_3 \sim v$, and $v_4 \in V_3$ by the choice of (C, P, v_1). If $u_3 \sim v_5$ then $v_5 \in V_3$ (by the choice of (C, P, v_1)) and $n \geq 4$ (as $d(u_n, C) \geq 2$); so $(u_3u_4, u_3v_5, u_3u_2v_g, u_3vw_1)$ would be a fork. Hence, $u_3 \nsim v_5$. Then $u \sim w_1$ to avoid $(vw_1, vu_1, vu_3u, vv_4v_5)$. Now $(w_1u, w_1x_1, w_1v_1v_g, w_1vv_4)$ is a fork, a contradiction.

By Claim 1 and Lemma 3.3, $N(x_1)\setminus\{w_1\} \nsubseteq N(x_2)\setminus\{v_2\}$ and $N(x_2)\setminus\{v_2\} \nsubseteq N(x_1)\setminus\{w_1\}$. Let $x \in N(x_2) \setminus (\{v_2\} \cup N(x_1))$.

Claim 2. $x \notin V(C)$.

For, assume $x \in V(C)$. Then $x = v_4$ by the choice of (C, P, v_1); so $v_4 \nsim x_1$. Hence, $u_3 \sim \{x_1, x_2\}$ to avoid $(u_1x_1, u_1v_1, u_1x_2v_4, u_1u_2u_3)$, and $u_3 \nsim x_1$ or $u_3 \nsim x_2$ to avoid $(u_3u, u_3x_1, u_3x_2v_4, u_3u_2v_g)$. If $u_3 \sim x_2$ and $u_3 \nsim x_1$, then $v_2, v_4 \in V_3$ by the choice of (C, P, v_1); so $(x_2v_2, x_2v_4, x_2u_1x_1, x_2u_3u)$ is a fork, a contradiction. Thus $u_3 \nsim x_2$ and $u_3 \sim x_1$.

Next, we show $x_2 \notin V_3$. For, assume $x_2 \in V_3$. Then $v_3 \in V_3$ by the choice of (C, P, v_1), and $v_4 \notin V_3$ by Corollary 4.3. Let $v \in N(v_4) \setminus \{v_3, v_5, x_2\}$. Since $og(G) \geq 7$ and because of the choice of (C, P, v_1), $v \notin V(C \cup P) \cup \{w_1, x_1\}$ and $v \nsim u_3$. Then $v \nsim w_1$ to avoid $(w_1v, w_1v_3, w_1x_1u_3, w_1v_1v_g)$, $v \sim u_1$ to avoid $(v_4v, v_4v_5, v_4v_3w_1, v_4x_2u_1)$, and $v \nsim v_6$ by the choice of (C, P, v_1). Hence, $(v_4v, v_4x_2, v_4v_5v_6, v_4v_3w_1)$ is a fork, a contradiction.

Thus, let $x' \in N(x_2) \setminus \{u_1, v_2, v_4\}$. Note that $x' \notin \{u_2, x_1\}$ since $og(G) \geq 7$, $x' \notin \{u_3, w_1\}$ since $x_2 \nsim \{u_3, w_1\}$, and $x' \notin V(C) \cup V(P - u_3)$ by the choice of (C, P, v_1). Now $x' \sim \{u_2, v_5\}$ to avoid $(x_2x', x_2v_2, x_2v_4v_5, x_2u_1u_2)$.

Suppose $x' \sim u_2$. Then $x' \nsim \{v_5, v_{g-1}\}$ by the choice of (C, P, v_1), and $x' \sim u$ or $u_3 \sim v_{g-1}$ to avoid $(u_2x', u_2u_1, u_2u_3u, u_2v_gv_{g-1})$. If $x' \sim u$ then $(x_2u_1, x_2v_2, x_2x'u, x_2v_4v_5)$ would be a fork. So $x' \nsim u$ and $u_3 \sim v_{g-1}$. Thus, $v_{g-1} \in V_3$ by the choice of (C, P, v_1), $u = u_4$ as $d(u_n, C) \geq 2$, and $u_4 \sim v_{g-2}$ to avoid $(u_3u_4, u_3u_2, u_3x_1w_1, u_3v_{g-1}v_{g-2})$. So $n \geq 5$ as $d(u_n, C) \geq 2$, and $(u_3v_{g-1}, u_3u_2, u_3u_4u_5, u_3x_1w_1)$ is a fork, a contradiction.

Hence, $x' \nsim u_2$, and $x' \sim v_5$. Let $x'' \in N(x') \setminus N(v_4)$. Since $og(G) \geq 7$ and because of the choice of (C, P, v_1), $x'' \notin V(C) \cup \{u_1, u_2, u_3\}$, $x'' \neq x_1$ as $x' \nsim x_1$, and $x'' \neq u$ to avoid $(x_2v_2, x_2v_4, x_2x'u, x_2u_1u_2)$. Now $x'' \sim \{u_1, v_2\}$ to avoid $(x_2v_2, x_2v_4, x_2u_1u_2, x_2x'x'')$. If $x'' \sim u_1$ then $x'' \sim w_1$ to avoid $(u_1x'', u_1u_2, u_1x_2v_4, u_1v_1w_1)$; hence $x'' \sim v_2$ by Claim 1, and $x'' \nsim u_3$ by the choice of (C, P, v_1); if $x'' \nsim u_1$ and $x'' \sim v_2$ then $x'' \sim w_1$ by Claim 1, and $x'' \nsim u_3$ by the choice of (C, P, v_1). So $(w_1x'', w_1v_3, w_1v_1v_g, w_1x_1u_3)$ is a fork, a contradiction.

Claim 3. $x_2 \nsim u_3$ (so $x \notin V(P)$ by the choice of (C, P, v_1)), and $x \nsim v_1$.

Suppose $x_2 \sim u_3$. Then $v_2 \in V_3$ by the choice of (C, P, v_1), and $w_1 \in V_3$ by Claim 1. Thus, replacing C, P with $v_1w_1v_3 \ldots v_g$, $v_1u_1x_2u_3 \ldots u_n$, respectively, we get a contradiction to Lemma 6.1 (since $v_1 \nsim \{v_3, v_{g-1}\}$ and $v_2 \sim x_2$).

Now assume $x \sim v_1$. Then $x \sim v_{g-1}$ to avoid $(v_1x, v_1v_2, v_1w_1x_1, v_1v_gv_{g-1})$, and $x \sim u_2$ to avoid $(v_1x, v_1v_2, v_1w_1x_1, v_1v_gu_2)$. Replacing P with $v_1xu_2 \dots u_n$, we get a contradiction to Lemma 5.2.

Claim 4. $x \nsim u_2$.

Suppose $x \sim u_2$. Then $x \nsim v_5$ by the choice of (C, P, v_1), and $x \sim u$ or $x_1 \sim u_3$ to avoid $(u_2x, u_2v_g, u_2u_3u, u_2u_1x_1)$. If $u = u_4$ and $x \sim u_4$ then $v_2 \in V_3$ by the choice of (C, P, v_1), and $w_1 \in V_3$ by Claim 1; hence replacing C, P with $v_1w_1v_3 \dots v_gv_1$, $v_1u_1x_2xu_4 \dots u_n$, respectively, we get a contradiction to Lemma 6.1 (since $u_1 \nsim \{v_3, v_{g-1}\}$ and $v_2 \sim x_2$). So $u \neq u_4$, or $u = u_4$ and $x \nsim u_4$.

We may thus choose u so that $u \nsim \{u_1, x\}$. For otherwise, $u \neq u_4$, $n = 3$, and, since $N(u_3) \nsubseteq N(x)$, there exists $u' \in N(u_3) \setminus (N(x) \cup \{u_1\})$ such that $u' \sim u_1$. Then $(u_1u', u_1x_1, u_1x_2x, u_1v_1v_g)$ is a fork, a contradiction.

Thus, $x_1 \sim u_3$ to avoid $(u_2v_g, u_2x, u_2u_3u, u_2u_1x_1)$, and $v_{g-1} \sim \{u_3, x\}$ to avoid $(u_2x, u_2u_1, u_2u_3u, u_2v_gv_{g-1})$. If $v_{g-1} \sim u_3$ then $v_{g-1} \in V_3$ by the choice of (C, P, v_1), and $u = u_4$ as $d(u_n, C) \geq 2$; so $(u_3v_{g-1}, u_3u_4, u_3u_2x, u_3x_1w_1)$ is a fork, a contradiction. Hence, $v_{g-1} \nsim u_3$ and $v_{g-1} \sim x$.

If $w_1 \notin V_3$ and let $w \in N(w_1) \setminus \{v_1, v_3, x_1\}$, then $w \sim v_2$ by Claim 1, and $w \nsim u_3$ by the choice of (C, P, v_1); so $(w_1w, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ would be a fork. Hence $w_1 \in V_3$, and then $v_2 \in V_3$ by Claim 1. So $v_3 \notin V_3$ by Corollary 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \notin V(C \cup P) \cup \{x\}$ by the choice of (C, P, v_1), and $v \notin \{x_1, x_2\}$ since $og(G) \geq 7$.

Suppose $v \sim x_2$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \sim x_1$ to avoid $(x_2v, x_2v_2, x_2xv_{g-1}, x_2u_1x_1)$. If $v \in V_3$ for all $v \in N(v) \setminus \{v_2, v_4, w_1\}$ then $v_3 \in V_4$ by Lemma 3.2, and hence v_4, v_3 contradict Lemma 3.4. So $v \notin V_3$ for some choice of v, and let $v' \in N(v) \setminus \{v_3, x_1, x_2\}$. Note that $v \nsim v_5$ (so $v' \neq v_5$) by the choice of (C, P, v_1), and $v' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3vv')$. Hence $v' \nsim x$ by the choice of (C, P, v_1), and $v' \sim u_1$ to avoid $(x_2v_2, x_2u_1, x_2xv_{g-1}, x_2vv')$. But then $(u_1u_2, u_1x_1, u_1v'v_4, u_1v_1v_2)$ is a fork, a contradiction.

So $v \nsim x_2$ for some $v \in N(v_3) \setminus \{v_2, v_4, w_4\}$. Then $v \sim v_5$ to avoid $(v_3w_1, v_3v, v_3v_2x_2, v_3v_4v_5)$ (as $x_2 \nsim v_4$ by Claim 2), and $x_1 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3u_2x_2, v_3w_1x_1)$. If $x_1 \sim v$ then let $v' \in N(v_4) \setminus N(v)$; now $(v_3v, v_3w_1, v_3v_2x_2, v_3v_4v')$ is a fork, a contradiction. If $x_1 \sim v_4$ then let $v' \in N(v) \setminus N(v_4)$; now $(v_3v_4, v_3w_1, v_3v_2x_2, v_3vv')$ is a fork, a contradiction.

By Claims 3 and 4, $x \nsim \{u_2, v_1\}$ and $x_2 \nsim u_3$. So $u_3 \sim x_1$ to avoid $(u_1v_1, u_1x_1, u_1u_2u_3, u_1x_2x)$. We claim that $w_1 \in V_3$. For, let $w \in N(w_1) \setminus \{v_1, v_3, x_1\}$. Then $w \notin V(C) \cup V(P - \{u_1, u_2\})$ by the choice of (C, P, v_1), $w \notin \{u_2, x_2\}$ as $w_1 \nsim \{u_2, x_2\}$, and $w \notin \{u_1, x\}$ as $og(G) \geq 7$. Since $w \sim v_2$ (by Claim 1), $w \nsim u_3$ by the choice of (C, P, v_1). So $(w_1w, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ is a fork, a contradiction.

Therefore, $v_2 \in V_3$ by Claim 1, and hence $v_3 \notin V_3$ by Corollary 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ be arbitrary. Then $v \notin V(C) \cup V(P - u_1)$ by the choice of (C, P, v_1), $v \notin \{x_1, x_2\}$ since $og(G) \geq 7$, and $v \neq u_1$ as $u_1 \nsim v_3$.

Suppose $v \sim x_1$ for some $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \nsim v_1$ to avoid $N(w_1) \subseteq N(v)$. Note that $v \sim v_5$; as otherwise replacing P with $v_3vx_1u_3 \dots u_n$ we get a contradiction to Lemma 6.1 (since $w_1 \sim x_1$). So $v_5 \nsim u$ by the choice of (C, P, v_1), and $v \sim u$ to avoid

$(x_1w_1, x_1u_1, x_1u_3u, x_1vv_5)$. Hence, $(vv_5, vu, vv_3v_2, vx_1u_1)$ is a fork, a contradiction.

So $v \nsim x_1$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. If $v_4 \sim x_1$ then $v_4 \in V_3$ by the choice of (C, P, v_1); so $(x_1u_1, x_1w_1, x_1u_3u, x_1v_4v_5)$ would be a fork. Hence $v_4 \nsim x_1$; so $v \sim v_5$ to avoid $(v_3v, v_3v_2, v_3w_1x_1, v_3v_4v_5)$, and $x_2 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3v_2x_2, v_3w_1x_1)$. If $x_2 \sim v$ then let $v' \in N(v_4) \setminus N(v)$; now $(v_3v, v_3v_2, v_3w_1x_1, v_3v_4v')$ is a fork, a contradiction. If $x_2 \sim v_4$ then let $v' \in N(v) \setminus N(v_4)$; now $(v_3v_4, v_3v_2, v_3w_1x_1, v_3vv')$ is a fork, a contradiction. ∎

Lemma 6.3. *$n \leq 2$.*

Proof. Suppose $n \geq 3$.

Case 1. $w_1 \sim \{v_3, v_{g-1}\}$.

Then by Lemma 6.2, $u_1 \sim \{v_3, v_{g-1}\}$. If $w_1 \sim v_3$ and $u_1 \sim v_{g-1}$ then by Lemma 5.6, $(N(v_g) \setminus N(u_1)) \cap (N(v_2) \setminus N(w_1)) = \emptyset$. Let $x \in N(v_2) \setminus N(w_1)$ and $y \in N(v_g) \setminus N(u_1)$; so $x \neq y$. By Lemma 5.5, $x \sim v_g$ or $y \sim v_2$. If $x \sim v_g$ then $x \sim u_1$ (since $x \notin N(v_g) \setminus N(u_1)$), contradicting Lemma 5.4 (as $x \nsim w_1$). So $y \sim v_2$. Hence $y \sim w_1$ (since $y \notin N(v_2) \setminus N(w_1)$), contradicting Lemma 5.4 again (as $y \nsim u_1$).

Similarly, if $w_1 \sim v_{g-1}$ and $u_1 \sim v_3$, we get a contradiction to Lemma 5.4. Thus by symmetry, we may assume that $v_3 \sim u_1$, $v_3 \sim w_1$ and $v_{g-1} \nsim \{u_1, w_1\}$. Then $v_2 \nsim P - v_1$ by Lemma 5.2. If $w_1 \nsim u_2$ then $u_2 \sim v_4$ to avoid $(v_3v_2, v_3w_1, v_3v_4v_5, v_3u_1u_2)$, and $u_2 \nsim v_g$ by the choice of (C, P, v_1); so $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$ is a fork, a contradiction. Thus $w_1 \sim u_2$, and hence u_1 and w_1 are symmetric. Let $u \in N(u_1) \setminus N(w_1)$ and $w \in N(w_1) \setminus N(u_1)$. Then $u, w \notin V(C \cup P)$ by the choice of (C, P, v_1).

Note that $w \sim \{v_2, v_4\}$ to avoid $(v_3v_2, v_3u_1, v_3v_4v_5, v_3w_1w)$, and $w \sim \{v_2, v_g\}$ to avoid $(v_1v_2, v_1u_1, v_1v_gv_{g-1}, v_1w_1w)$. Hence, $w \sim v_2$, since $w \nsim v_4$ or $w \nsim v_g$ (by the choice of (C, P, v_1)). Similarly, $u \sim v_2$. Thus, by the choice of (C, P, v_1), $u_3 \nsim \{u, w\}$, and $\{u, w\} \sim \{v_4, v_g\}$ to avoid $(v_2u, v_2w, v_2v_3v_4, v_2v_1v_g)$. So by symmetry let $w \sim v_4$; then $w \nsim v_g$ by the choice of (C, P, v_1).

Now $v_g \sim u_2$ to avoid $(w_1w, w_1v_3, w_1u_2u_3, w_1v_1v_g)$. Thus, $v_g \in V_3$ by the choice of (C, P, v_1), and $u_3 \sim v_{g-1}$ to avoid $(u_2u_3, u_2u_1, u_2w_1w, u_2v_gv_{g-1})$. So $n \geq 4$ as $d(u_n, C) \geq 2$, and $(u_2u_1, u_2v_g, u_2u_3u_4, u_2w_1w)$ is a fork in G, a contradiction.

Case 2. $w_1 \nsim \{v_3, v_{g-1}\}$.

Then $u_1 \sim \{v_3, v_{g-1}\}$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_gv_{g-1})$. If $w_1 \sim u_2$ then by replacing P with $v_1w_1u_2 \ldots u_n$ we get back to Case 1. So $w_1 \nsim u_2$. By symmetry assume $u_1 \sim v_{g-1}$.

Then $v_g \nsim P - v_1$ by Lemma 5.2. So $u_2 \sim v_2$ to avoid $(v_1v_g, v_1w_1, v_1v_2v_3, v_1u_1u_2)$. Hence, $v_2 \in V_3$ by the choice of (C, P, v_1), and $\{u_1, u_2\} \nsubseteq V_3$ by Corollary 4.3. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Note that $u \notin V(C)$, and $u \neq w_1$ by the choice of (C, P, v_1).

Subcase 2.1. $u_2 \notin V_3$.

Let $u' \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then, since $u_2 \nsim \{v_g, w_1\}$ and $og(G) \geq 7$, $u' \notin V(C \cup P) \cup \{u, w_1\}$ by the choice of (C, P, v_1). So $u' \sim \{u, v_{g-1}\}$ to avoid $(u_2u', u_2v_2, u_2u_3u, u_2u_1v_{g-1})$. Note that $u' \nsim v_{g-1}$; otherwise, replacing P with $v_{g-1}u'u_2 \cdots u_n$, we get back to Case 1. So $u' \sim u$ (for all choice of u). Now $u' \nsim v_1$; otherwise replacing P with $v_1u'uu_3 \ldots u_n$ we get back to Case 1.

We claim $u = u_4$. For, otherwise, since $N(u_3) \not\subseteq N(u')$, there exists $u'' \in (N(u_3 \setminus N(u'))) \cap N(u_1)$. Note that $w_1 \nsim u''$; otherwise replacing P with $v_1 w_1 u'' u_3 \dots u_n$ we get back to Case 1. Hence $(u_1 u'', u_1 v_{g-1}, u_1 u_2 u', u_1 v_1 w_1)$ is a fork, a contradiction.

So we have symmetry between u' and u_3, and thus we also have $u_3 \nsim v_{g-1}$. Hence, $v_3 \sim \{u', u_3\}$ to avoid $(u_2 u', u_2 u_3, u_2 u_1 v_{g-1}, u_2 v_2 v_3)$. By symmetry, let $u' \sim v_3$. Then $v_3 \in V_3$ by the choice of (C, P, v_1). Let $u_3' \in N(u_3) \setminus N(u')$. Then by the choices of (C, P, v_1) and the assumption $og(G) \geq 7$, $u_3' \notin V(C \cup P) \cup \{w_1\}$. Now $u_3' \sim u_1$ to avoid $(u_2 u', u_2 v_2, u_2 u_3 u_3', u_2 u_1 v_{g-1})$, and $u_3' \sim w_1$ to avoid $(u_1 u_3', u_1 v_{g-1}, u_1 u_2 u', u_1 v_1 w_1)$. But then, replacing P with $v_1 w_1 u_3' u_3 \cdots u_n$, we get back to Case 1.

Subcase 2.2. $u_2 \in V_3$ and $u_1 \notin V_3$.

Since $u_2 \in V_3$, $v_g \in V_3$; otherwise the cycle $v_2 v_3 \dots v_{g-1} u_1 u_2 v_2$ contradicts the choice of C in (C, P, v_1). Let $u' \in N(u_1) \setminus \{u_2, v_1, v_{g-1}\}$. Then $u' \notin V(C \cup P)$ by the choice of (C, P, v_1), $u' \neq u$ as $u \nsim u_1$, $u' \neq w_1$ as $og(G) \geq 7$, $u' \sim \{u_3, v_{g-2}\}$ to avoid $(u_1 u', u_1 v_1, u_1 u_2 u_3, u_1 v_{g-1} v_{g-2})$, $u' \nsim v_g$ to avoid $N(v_g) \subseteq N(u_1)$, and $u' \sim w_1$ to avoid $(v_1 w_1, v_1 v_g, v_1 v_2 v_3, v_1 u_1 u')$. So $u' \nsim u_3$; otherwise, replacing P with $v_1 w_1 u u_3 \dots u_n$, we get back to Case 1. Thus, $u' \sim v_{g-2}$.

We claim that $u' \in V_3$. For, suppose there exists $u'' \in N(u') \setminus \{u_1, v_{g-2}, w_1\}$. Then $u'' \neq u$ since $og(G) \geq 7$, $u'' \notin V(C - v_g) \cup V(P - u_3)$ by the choice of (C, P, v_1), $u'' \notin \{u_3, v_g\}$ as $u' \nsim \{u_3, v_g\}$, and $u'' \sim \{v_1, v_{g-1}\}$ to avoid $(u_1 v_1, u_1 v_{g-1}, u_1 u_2 u_3, u_1 u' u'')$. If $u'' \sim v_1$ then $u'' \nsim v_{g-3}$ by the choice of (C, P, v_1), and $u'' \sim v_{g-1}$ to avoid $(v_1 u'', v_1 w_1, v_1 v_2 v_3, v_1 v_g v_{g-1})$; so $(v_{g-1} u'', v_{g-1} v_g, v_{g-1} v_{g-2} v_{g-3}, v_{g-1} u_1 u_2)$ is a fork, a contradiction. So $u'' \nsim v_1$ and $u'' \sim v_{g-1}$. Then $u'' \sim v_{g-3}$ to avoid $(v_{g-1} u'', v_{g-1} v_g, v_{g-1} v_{g-2} v_{g-3}, v_{g-1} u_1 u_2)$. Now, replacing P with $v_{g-1} u_1 u_2 \cdots u_n$, we get back to Case 1.

Thus $v_{g-1} \in V_3$ as otherwise $v_2 \dots v_{g-2} u' u_1 u_2 v_2$ would contradict the choice of C in (C, P, v_1). Hence, $v_{g-2} \notin V_3$ by Corollary 4.3. Let $v \in N(v_{g-2}) \setminus \{v_{g-1}, v_{g-3}, u'\}$. Then $v \notin V(C) \cup V(P - u_1)$ by the choice of (C, P, v_1), $v \notin \{u_1, w_1\}$ since $og(G) \geq 7$, $v \sim \{v_{g-4}, v_g\}$ to avoid $(v_{g-2} v, v_{g-2} u', v_{g-2} v_{g-1} v_g, v_{g-2} v_{g-3} v_{g-4})$, and $v \sim \{v_g, w_1\}$ to avoid $(v_{g-2} v, v_{g-2} v_{g-3}, v_{g-2} u' w_1, v_{g-2} v_{g-1} v_g)$. Hence, $v \sim v_g$ (for any choice of v); otherwise, $v \sim v_{g-4}$ and $v \sim w_1$, contradicting the choice of (C, P, v_1). Thus, $v_{g-2} \in V_4$ as $v_g \in V_3$, and $v \sim \{u_1, w_1\}$ to avoid $(v_1 u_1, v_1 w_1, v_1 v_2 v_3, v_1 v_g v)$. But $v \nsim u_1$ to avoid $N(v_{g-1}) \subseteq N(v)$; so $v \sim w_1$.

If $v \in V_3$ then v_{g-3}, v_{g-2} contradict Lemma 3.4. So $v \notin V_3$, and let $v' \in N(v) \setminus \{v_{g-2}, v_g, w_1\}$. Then $v' \notin V(C) \cup V(P - \{u_1, u_2\})$ by the choice of (C, P, v_1), $v' \notin \{u', u_2\}$ as $og(G) \geq 7$, and $v' \neq u_1$ as $v \nsim u_1$. So $v' \sim v_{g-3}$ to avoid $(v_{g-2} u', v_{g-2} v_{g-1}, v_{g-2} v v', v_{g-2} v_{g-3} v_{g-4})$. Now $v \in V_4$; for otherwise, let $v'' \in N(v) \setminus \{v_{g-2}, v_g, v', w_1\}$, then $(v v', v v'', v v_g v_{g-1}, v w_1 u')$ is a fork, a contradiction.

Suppose there exists $w \in N(w_1) \setminus \{u', v, v_1\}$. Then, since $w_1 \nsim \{u_2, v_3, v_{g-1}\}$ and $og(G) \geq 7$, $w \notin V(C \cup P)$ by the choice of (C, P, v_1). So $w \sim u_1$ to avoid $(v_1 u_1, v_1 v_g, v_1 v_2 v_3, v_1 w_1 w)$, and $w \sim u_3$ to avoid $(u_1 w, u_1 u', u_1 v_1 v_g, u_1 u_2 u_3)$. Now, replacing P with $v_1 w_1 w u_3 \cdots u_n$, we get back to Case 1.

Thus $w_1 \in V_3$. By the choice of G, $G - \{u', v, v_{g-2}, v_{g-1}, v_g, w_1\}$ has a 3-coloring c. By setting $c(v_{g-2}) = c(v')$ and greedily coloring $\{u', v, v_{g-1}\}$ (with a common color for all three), v_g, w_1 in order, we get a 3-coloring of G, a contradiction. ∎

7 Conclusion

Proof. We complete now the proof of Theorem 1.2. Let $C = v_1 \dots v_g v_1$ be a shortest cycle in G such that the assertion of Lemma 6.3 holds. By Corollary 2.2, we see that $N_2(C) \neq \emptyset$. Let $T \subseteq V(G - C)$ such that for any $u \in T$ if P is a path in G from u to some v_i with $V(P) \cap V(C) = \{v_i\}$ then $v_i \in V_3$, and let H be the union of all such paths P from T to C. Define $S = V(H) \cap V(C)$; so $S \subseteq V_3$. Let $K = G - (H - S)$. Note that $S \subseteq V_2(K)$.

By Corollary 4.5, $S \neq V(C)$. So by the choice of G, $\chi(H) \leq 3$. Let c_H be a 3-coloring of H, which induces a 3-coloring c_S on $G[S]$. If the conditions of Lemma 2.1 hold, then by Lemma 2.1, c_S can be extended to a 3-coloring of c_K of K; now let $c(v) = c_K(v)$ if $v \in V(K)$ and $c(v) = c_H(v)$ if $v \in V(H)$, we see that c is a 3-coloring of G, a contradiction. Thus, it suffices to verify the conditions of Lemma 2.1. Recall the notation in Section 2.

By Lemma 6.3, we see that if $u \in N_2(C) \cap V(K)$ then there is a path uu_1v_i in K such that $v_i \notin V_3$. Let $w_1 \in N(v_i) \setminus \{u_1, v_{i-1}, v_{i+1}\}$. Then $\{u_1, w_1\} \sim \{v_{i-2}, v_{i+2}\}$ to avoid $(v_iu_1, v_iw_1, v_iv_{i-1}v_{i-2}, v_iv_{i+1}v_{i+2})$. By symmetry and by the minimality of C, assume $w_1 \nsim v_{i+2}$. If $u_1 \sim \{v_{i-2}, v_{i+2}\}$ then u is associated with v_{i-1} or v_{i+1}. On the other hand, if $u_1 \nsim \{v_{i-2}, v_{i+2}\}$ then $u \sim w_1$ to avoid $(v_iw_1, v_iv_{i-1}, v_iv_{i+1}v_{i+2}, v_iu_1u)$, and $w_1 \sim v_{i-2}$ to avoid $(v_iw_1, v_iv_{i+1}, v_iu_1u, v_iv_{i-1}v_{i-2})$; so u is associated with v_{i-1}.

So in K, every vertex in $N_2(C) \cap V(K)$ is associated with a vertex of C. Next we show that (i) – (iii) of Lemma 2.1 holds.

Suppose $w \in N(v_i)$ and $x_1, x_2 \in N(w) \cap N_2(C)$, such that x_1 is associated with one of $\{v_{i-3}, v_{i-1}\}$ and x_2 is associated with one of $\{v_{i+3}, v_{i+1}\}$. We show that $v_{i-1}, v_{i+1} \notin S$. By the minimality of C and by symmetry we may assume x_1 is associated with v_{i-1}. Let $x_1u_1v_i$ and $x_1u_1v_{i-2}$ be paths in K. Note that $u_1 \nsim x_2$ to avoid $(u_1x_1, u_1x_2, u_1v_{i-2}v_{i-3}, u_1v_iv_{i+1})$. If $v_{i+1} \in S$ then let $v \in N(v_{i+1}) \setminus \{v_i, v_{i+2}\}$; now $(v_iu_1, v_iv_{i-1}, v_iwx_2, v_iv_{i+1}v)$ is a fork, a contradiction. If $v_{i-1} \in S$ then let $v \in N(v_{i-1}) \setminus \{v_i, v_{i-2}\}$; now $(v_iu_1, v_iv_{i+1}, v_iwx_2, v_iv_{i-1}v)$ is a fork, a contradiction. So $v_{i-1}, v_{i+1} \notin S$, and Lemma 2.1(i) holds.

Now suppose $X_{i,1} \neq \emptyset$ and some $v_j \in \{v_{i-1}, v_{i+1}\} \cap S$ is associated with some vertex $u \in N_2(C)$. Without loss of generality, let $v_j = v_{i+1}$, and let uu_1v_i and uu_1v_{i+2} be paths. Let $w \in X_{i,1}$ and $v \in N(v_{i+1}) \setminus \{v_i, v_{i+2}\}$. Then $(v_iw, v_iu_1, v_iv_{i-1}v_{i-2}, v_iv_{i+1}v)$ is a fork, a contradiction. So Lemma 2.1(ii) holds.

Finally, we verify Lemma 2.1(iii). By symmetry, assume that v_i is associated with some vertex $u \in N_2(C)$ and $u \sim w \in X_{i+1,1} \cup X^+_{i+1,2} \cup X^+_{i+1,3}$. Suppose $v_i, v_{i+3} \in S$. Then $w \nsim v_{i+3}$. Let $v \in V(H) \setminus V(K)$. Let uu_1v_{i-1} and uu_1v_{i+1} be paths. Then $(v_{i+1}u_1, v_{i+1}w, v_{i+1}v_{i+2}v_{i+3}, v_{i+1}v_iv)$ is a fork, a contradiction. So Lemma 2.1(iii) holds. ∎

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