Non-Separating Cycles in 4-Connected Graphs

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Abstract

We prove that given any fixed edge ra in a 4-connected graph G, there exists a cycle C through ra such that $G - (V(C) - \{r\})$ is 2-connected. This will provide the first step in a decomposition for 4-connected graphs. We also prove that for any given edge e in a 5-connected graph G there exists an induced cycle C through e in G such that $G - V(C)$ is 2-connected. This provides evidence for a conjecture of Lovász.

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1 Introduction and notation

Throughout the paper, we consider only simple graphs. We let $G = (V(G), E(G))$ be the graph with *vertex set* $V(G)$ and *edge set* $E(G)$. We use the shorthand notation xy (or yx) for an edge in $E(G)$ whose ends are x and y. For two subgraphs G and H of a graph, we use $G \cup H$ and $G \cap H$ to denote their union and intersection respectively. For convenience, we use $A := B$ to rename B as A or to define A as B.

Let G be a graph. Given $x \in V(G)$, let $N_G(x) := \{y \in V(G) : yx \in E(G)\}$. Given $S \subseteq V(G)$, we let $N_G(S) := \{x \in V(G) - S : xy \in E(G) \text{ for some } y \in S\}$. For a subgraph H of G, we write $N_G(H) := N_G(V(H))$. When ambiguity is not a concern, we may simply use $V, E, N(x), N(S)$ and $N(H)$. Let P be a path between vertices u and v in G; then P is called a $u - v$ path, and u and v are called the *ends* of P. Given vertices x, y on P, we let xPy denote the path in P with ends x and y. Let X be a set of 2-element subsets of $V(G)$; then $G + X$ will denote the graph with vertex set $V(G)$ and edge set $E(G) \cup X$.

Given $S \subseteq V(G)$, $G[S]$ will denote the subgraph of G induced by S, and let $G - S := G[V(G) - S]$. For $S \subset E(G)$, we let $G - S$ denote the graph obtained from G by deleting edges in S. If $S = \{s\} \subset V(G) \cup E(G)$, we let $G - s := G - S$. A cycle C in G is an *induced* cycle if $G[V(C)] = C$, and it is *non-separating* if $G - V(C)$ is connected.

A *plane graph* is a graph which is drawn in the plane with no pair of edges crossing. The *faces* of a plane graph are the connected components (in topological sense) of its complement in the plane. The *infinite face* of a plane graph is its unbounded face. The boundary of a face is called a *facial walk*, or *facial cycle* if it is a cycle. A graph is *planar* if it is isomorphic to a plane graph.

An *ear decomposition* of a connected graph G is a set $\mathcal{E}_G = \{P_0, P_1, P_2, \ldots, P_k\}$ which satisfies the following three conditions: (1) P_0 is a cycle in G; (2) if $1 \leq i \leq k$, then P_i is a path in G with ends $\{x, y\}$ such that $\left(\bigcup_{j=0}^{i-1} E(P_j)\right) \cap E(P_i) = \emptyset$ and $\left(\bigcup_{j=0}^{i-1} V(P_j)\right) \cap V(P_i) = \{x, y\}$; and (3) $G = \left(\bigcup_{j=0}^{k} V(P_j),\ \bigcup_{j=0}^{k} E(P_j)\right)$. The elements of $\mathcal{E}_G$ are called *ears* of G.

Let $T_1, T_2, \ldots, T_m$ be spanning trees of a graph G and let $r \in V(G)$. We say that $T_1, \ldots, T_m$ are *independent spanning trees of G rooted at r* if for any $x \in V(G)$ and for any distinct $i, j \in \{1, \ldots, m\}$ the $r - x$ paths in T_i and T_j are vertex-disjoint in G except at r and x. Given any vertex r in a 2-connected graph G, it is known that G contains two independent spanning trees rooted at r; Itah and Rodeh [5] constructed these trees using an ear decomposition of G. In [12], Itah and Zehavi showed that if G is a 3-connected graph and $r \in V(G)$, then G contains three independent spanning trees rooted at r. Their proof relied on the property that every

3-connected graph with at least five vertices contains a *contractible* edge - one whose contraction results in a new 3-connected graph. Since this property is unique to 3-connected graphs, there is little hope of generalizing their approach to cases with higher connectivity. Cheriyan and Maheshwari [3] independently showed the 3-connected result; however, they used an ear decomposition of the graph, albeit a more restrictive type called a *non-separating* ear decomposition. This non-separating ear decomposition $\{P_0, P_1, \ldots\}$ imposes connectivity conditions between P_i and $G - \left(\bigcup_{j=1}^{i} V(P_j)\right)$ and also on $G - \left(\bigcup_{j=1}^{i} V(P_j)\right)$. The first ear P_0 of this decomposition is guaranteed by the following result of Tutte [11].

Theorem 1.1. *Let G be a 3-connected graph, let $st \in E(G)$, and let $r \in V(G)$ such that $r \notin \{s,t\}$. Then G contains a non-separating induced cycle through st and avoiding r.*

In [12], it is conjectured that for any vertex r in a k-connected graph G, there exist k independent spanning trees of G rooted at r. The 4-connected case is very interesting, because it is the first case where the existence of a contractible edge is not guaranteed. A. Huck [4] has shown that every 4-connected planar graph contains four independent spanning trees rooted at any given vertex. We would like to devise a 4-connected version of the non-separating ear decomposition which could be used to construct four independent spanning trees (rooted at any given vertex r) in 4-connected graphs. The first step in building such a decomposition is to find a cycle C through the "root" r which leaves a high degree of connectivity in $G - (V(C) - \{r\})$. Our construction of such a cycle is the main result of this paper.

Theorem 1.2. *Let G be a 4-connected graph, and let $ra \in E(G)$. Then G contains a cycle C through ra such that $G - (V(C) - \{r\})$ is 2-connected.*

While motivated by the search for an ear decomposition, this result is independently interesting. For example, variations of our proof give the following two results.

Theorem 1.3. *Let G be a 5-connected graph, and let $e \in E(G)$. Then G contains an induced cycle C through e such that $G - V(C)$ is 2-connected.*

Theorem 1.4. *Let G be a planar 4-connected graph, and let C be a non-separating induced cycle in G. Then for any $r \in V(C)$, $G - (V(C) - \{r\})$ is 2-connected.*

Note that Theorem 1.3 closely parallels Theorem 1.1, and a 6-connected version was shown by Kriesell [6]. As a consequence of Theorem 1.3, we can deduce the following result (proved independently in [6] and [2]): for any 5-connected graph G and $\{a, b\} \subset V(G)$, G contains an induced $a - b$ path P such that $G - V(P)$ is 2-connected. This result in turn provides some evidence for the following conjecture of Lovász [7]: Given any positive integer k, there exists some positive integer $f(k)$ with the property that for any given vertices x and y in a $f(k)$-connected graph G, there exists an induced $x-y$ path P in G such that $G-V(P)$ is k-connected.

We note that a cycle in a 3-connected plane graph is non-separating and induced if and only if it is a facial cycle. Therefore, Theorem 1.4 says that if G is a 4-connected plane graph and C is any facial cycle of G, then for each $r \in V(C)$, $G-(V(C)-\{r\})$ is 2-connected.

Our paper will progress as follows: In Section 2, we establish some convenient definitions and state some known results. Three technical lemmas will be shown: two are deduced from well-known results on paths in graphs, and the last is an independent lemma necessary for proving Theorems 1.2 and 1.3. In Section 3, we prove Theorem 1.2; in fact, we will prove a stronger result, Theorem 3.1. Its proof constructs a non-separating cycle C for Theorem 1.2, and reveals some structure which will be useful in constructing non-separating ear decompositions of 4-connected graphs. In Section 4, we modify our proof of Theorem 3.1 to deduce Theorem 1.3. We also prove Theorem 1.4. In Section 5, we offer some concluding remarks.

2 Preliminary results

For notational convenience, we begin this section with the following definition. Let G be a graph with distinct vertices a, b, c, and d. We say that the ordered quintuple (G, a, b, c, d) is *planar* if G can be drawn in a closed disc in the plane with no pair of edges crossing such that a, b, c, d occur on the boundary of the disc in this cyclic order.

Establishing planarity of certain subgraphs will be critical in the proof of Theorem 3.1 in Section 3. To this end, we use a well-known result of Seymour [8]. Different versions of this result were obtained independently by Chakravarti and Robertson [1] and by Thomassen [9].

Theorem 2.1. *Let u_1, v_1, u_2, v_2 be distinct vertices of a graph $G=(V,E)$. Then exactly one of the following is true:*

(1) there are vertex disjoint paths joining u_1 to v_1 and u_2 to v_2 respectively.

(2) for some integer $k \geq 0$ there are pairwise disjoint sets $A_1, A_2, \ldots, A_k \subseteq V-\{u_1, u_2, v_1, v_2\}$ such that

(a) for $1 \leq i \neq j \leq k$, $N(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$,

(c) if G' is the graph obtained from G by, for each i, deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, and also for $j=1,2$ adding an edge e_j joining u_j to

v_j, then G' may be drawn in the plane with no pairs of edges crossing except e_1, e_2, which cross once.

The following corollary is a simpler version of Theorem 2.1, attained by imposing some connectivity conditions.

Corollary 2.2. *Let $u_1,\ u_2,\ v_1,\ v_2$ be distinct vertices of a graph G. Suppose that for any $T \subset V(G)$ with $|T| \leq 3$, every component of $G - T$ contains at least one element of $\{u_1,\ u_2,\ v_1,\ v_2\}$. Then exactly one of the following is true:*

(1) there are vertex disjoint paths joining u_1 to v_1 and u_2 to v_2 respectively.

(2) $(G, u_1,\ u_2,\ v_1,\ v_2)$ is planar.

Proof. Clearly, (1) and (2) are mutually exclusive because of planarity. We know that either (1) or (2) of Theorem 2.1 must hold. If (1) of Theorem 2.1 holds, then (1) of Corollary 2.2 holds. So assume (2) of Theorem 2.1 holds. Then $\{u_1,\ u_2,\ v_1,\ v_2\} \cap A_i = \emptyset$ for all $1 \leq i \leq k$. Hence $G[A_i]$ consists of those components of $G - N(A_i)$ containing no element of $\{u_1,\ u_2,\ v_1,\ v_2\}$, contradicting our hypothesis. So no A_i may exist. Let G', e_1 and e_2 be described as in (c) of Theorem 2.1. Observe that $(G' - \{e_1, e_2\}, u_1, u_2, v_1, v_2)$ is planar. But $G' - \{e_1, e_2\} = G$. $\square$

Let P be a subgraph of G. Then a *P-bridge* of G is a subgraph of G which is induced by either (1) an edge in $E(G) - E(P)$ with both ends on P or (2) edges of a component of $G - V(P)$ and edges of G from that component to P. For any P-bridge B of G, the set $V(B \cap P)$ is the set of *attachments* of B on P.

In the proof of Theorem 3.1, we will reroute paths through planar subgraphs. To this end, we need a well-known theorem of Thomassen [10].

Theorem 2.3. *Let G be a 2-connected plane graph, F be a facial cycle of G, $x \in V(F)$, $e \in E(F)$, $y \in V(G) - \{x\}$. Then G contains an $x - y$ path P through e such that*

(1) every P-bridge of G has at most three attachments on P, and

(2) every P-bridge of G containing an edge of F has two attachments on P.

Note that if G is 4-connected and $|V(P)| \geq 4$, then P is a Hamilton path in G. We will apply Theorem 2.3 to certain planar subgraphs of a 4-connected graph. Therefore, it will be convenient to have the following corollary.

Corollary 2.4. *Let (G, a, c, b, d) be planar such that $G - \{c, d\}$ contains an $a - b$ path. Assume that for any $T \subset V(G)$ with $|T| \leq 3$, every component of $G - T$ must contain an element of $\{a, c, b, d\}$. Then $G - \{c, d\}$ contains an $a - b$ Hamilton path.*

Proof. Let $G' := (G - d) + \{\{b, c\}, \{a, c\}\}$. We first show that G' is 2-connected. Suppose on the contrary that G' is not 2-connected. Let x be a cut vertex of G'. Because $G - \{c, d\}$ contains an $a - b$ path, $\{a, b, c\}$ is contained in a cycle of G'. Therefore, $\{a, b, c\}$ is contained in an x-bridge of G', and G' has another x-bridge B such that $(V(B) - \{x\}) \cap \{a, b, c\} = \emptyset$. Hence, $B - x$ is a component of $G - T$, where $T := \{x, d\}$, and $V(B - x) \cap \{a, b, c, d\} = \emptyset$, a contradiction.

Observe that G' is planar, and may be drawn in the plane so that ac, bc, and $N(d)$ are on the cycle F which bounds its infinite face. Applying Theorem 2.3 (with G', a, c, bc as G, x, y, e, respectively), G' has an $a - c$ path P through bc satisfying (1) and (2) of Theorem 2.3. Note that $ac \notin E(P)$ because $bc \in E(P)$.

We proceed to show that every P-bridge of G' is induced by a single edge, and so P must be a Hamilton path in G'. Let B be a P-bridge of G' such that $V(B) - V(P) \neq \emptyset$, and let $T := V(B) \cap V(P)$. Since a, b, and c are all on P, then $\{a, b, c\} \cap V(B) \subseteq T$. Thus $B - T$ is a component of $G - (\{d\} \cup T)$ containing no element of $\{a, b, c, d\}$. If $|T| \leq 2$, then $|\{d\} \cup T| \leq 3$, contradicting our hypothesis. Since P must satisfy (1) of Theorem 2.3, we may assume $|T| = 3$. Then by (2) of Theorem 2.3, $E(B) \cap E(F) = \emptyset$, and hence $(V(B) - T) \cap N(d) = \emptyset$. Therefore, $B - T$ is a component of $G - T$ such that $V(B - T) \cap \{a, b, c, d\} = \emptyset$, a contradiction.

Thus $P - c$ is an $a - b$ Hamilton path in $G - \{c, d\}$, as required. □

Finally, we prove the following important technical lemma. We rely heavily on this result in the proof for Theorems 1.3 and 3.1.

Lemma 2.5. *Let G be a connected graph, let $S \subset V(G)$, and let $a, a', b, b' \in S$. Suppose*

(i) G contains vertex disjoint paths joining a to a' and b to b' respectively, and

(ii) for any $T \subseteq V(G)$ with $|T| \leq 2$, every component of $G - T$ contains a vertex of S.

Then $G - \{b, b'\}$ contains an induced $a - a'$ path P such that

(1) $\{b, b'\}$ is contained in a component of $G - V(P)$, and

(2) every component of $G - V(P)$ contains an element of S.

Proof. Let $\mathcal{P}$ be the set of those induced $a - a'$ paths P in $G - \{b, b'\}$ such that $\{b, b'\}$ is contained in a component of $G - V(P)$. By (i), $\mathcal{P} \neq \emptyset$. For each $P \in \mathcal{P}$, let B_P denote the component of $G - V(P)$ containing $\{b, b'\}$, and let T_P denote the union of those components C of $G - V(P)$ such that $V(C) \cap S = \emptyset$.

Select $P \in \mathcal{P}$ such that (a) $|V(B_P)|$ is maximum, and then (b) $|V(T_P)|$ is minimum. If $|V(T_P)| = 0$, then Lemma 2.5 holds. So assume $|V(T_P)| \neq 0$.

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be the set of components of $G - V(P)$ such that $C_i \subseteq T_P$. For $i = 1, \ldots, n$, we let a_i and a'_i be the elements of $N(C_i) \cap V(P)$ such that $a_i P a'_i$ is maximal. Let the notation be chosen so that a, a_i, a'_i, a' occur on P in the order listed. Let $\mathcal{K}$ be the auxiliary graph such that $V(\mathcal{K}) = \mathcal{C}$, and $C_i C_j \in E(\mathcal{K})$ if and only if $E(a_i P a'_i) \cap E(a_j P a'_j) \neq \emptyset$. Let $\mathcal{F}$ be a component of $\mathcal{K}$. From construction, $Q := \bigcup_{C_i \in V(\mathcal{F})} a_i P a'_i$ is a subpath of P. Let x and y be the ends of Q. See Figure 2 for an illustration.

Note that $V(Q) \neq \{x, y\}$, and there must exist some component K of $G - V(P)$ such that $V(K) \cap S \neq \emptyset$ and $N(K) \cap (V(Q) - \{x, y\}) \neq \emptyset$. Otherwise, the subgraph H of G induced by $\left(\bigcup_{C_i \in V(\mathcal{F})} V(C_i)\right) \cup (V(Q) - \{x, y\})$ is a union of components of $G - \{x, y\}$. But H contains no element of S, so $T := \{x, y\}$ violates hypothesis (ii).

Let $z \in N(K) \cap (V(Q) - \{x, y\})$. Then there exists some $C_i \in V(\mathcal{F})$ such that $z \in V(a_i P a'_i) - \{a_i, a'_i\}$. Otherwise, for any $C_j \in V(\mathcal{F})$, either $\{a_j, a'_j\} \subseteq V(xPz)$ or $\{a_j, a'_j\} \subseteq V(zPy)$. Let $\mathcal{F}_x$ be the subgraph of $\mathcal{F}$ induced by those C_j such that $\{a_j, a'_j\} \subseteq V(xPz)$, and let $\mathcal{F}_y$ be the subgraph of $\mathcal{F}$ induced by those C_j such that $\{a_j, a'_j\} \subseteq V(zPy)$. Then for any $C_k \in V(\mathcal{F}_x)$ and $C_l \in V(\mathcal{F}_y)$, $E(a_k P a'_k) \cap E(a_l P a'_l) = \emptyset$. Hence, $\mathcal{F}$ is not connected, a contradiction.

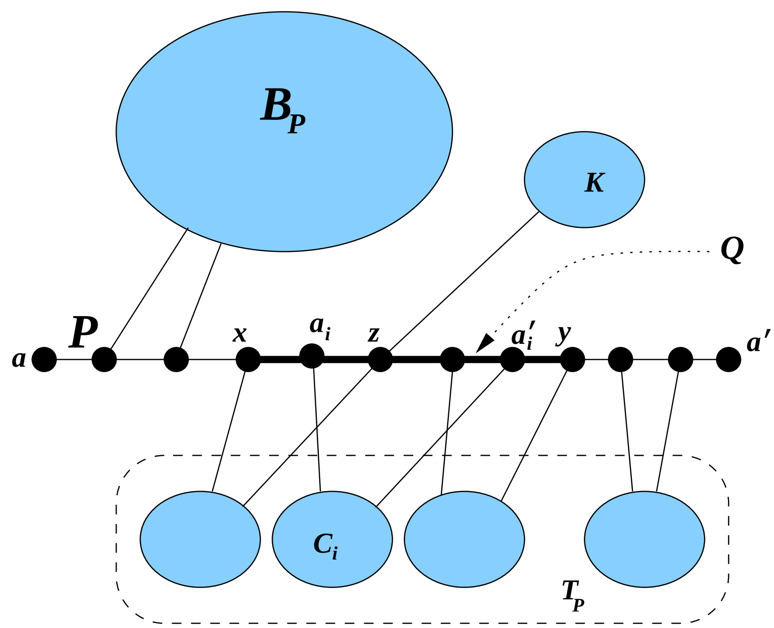

Figure 1: Lemma 2.5

Choose any induced $a_i - a'_i$ path R in $G[V(C_i) \cup \{a_i, a'_i\}]$, and let $X := aPa_i \cup R \cup a'_i P a'$. Clearly, X is an induced $a - a'$ path in G, and B_P is contained in a component of $G - V(X)$. Hence $X \in \mathcal{P}$ and $V(B_P) \subset V(B_X)$. But $V(T_X) \subset V(T_P) - V(R \cap C_i)$, contradicting (a) or (b). □

3 4-Connected Graphs

We prove our main result in this section. For the sake of the proof and for the application to independent trees (as described in Section 1), we prove the following stronger result.

Theorem 3.1. *Let G be a 4-connected graph, let $r \in V(G)$, and let $e \in E(G)$ such that e is incident with r. Then there exists a cycle C through e in G such that $G - (V(C) - \{r\})$ is 2-connected. Moreover, for some integer $m \geq 0$, there exist edge-disjoint subpaths P_t of $C - r$ with ends a_t and b_t, $1 \leq t \leq m$, such that*

(i) every chord of C has both ends on some P_t, and

(ii) for each $t \in \{1, \dots, m\}$, there exist distinct $c_t, d_t \in V(G) - V(C)$ such that $G[V(P_t) - \{a_t, b_t\}]$ is a component of $G - \{a_t, b_t, c_t, d_t\}$, and $(G[V(P_t) \cup \{c_t, d_t\}], a_t, c_t, b_t, d_t)$ is planar.

Proof. Let $\mathcal{D}$ denote the set of those induced cycles D in G such that $e \in E(D)$, $G-(V(D)-\{r\})$ is connected, and r is not a cut vertex of $G - (V(D) - \{r\})$.

By Theorem 1.1, G contains a non-separating induced cycle D through e. Since G is 4-connected, r must have at least four neighbors, and since D is induced, exactly two of those neighbors lie on D. Thus, $G-(V(D)-\{r\})$ is connected. Further, since $G-V(D)$ is connected, r is not a cut vertex of $G - (V(D) - \{r\})$. Hence $\mathcal{D} \neq \emptyset$.

For each $D \in \mathcal{D}$, let B_D denote the block of $G - (V(D) - \{r\})$ containing r. Since r is not a cut vertex of $G - (V(D) - \{r\})$, then $|V(B_D)| \geq 3$, and so B_D is 2-connected.

(a) We choose $D \in \mathcal{D}$ so that $|V(B_D)|$ is maximum.

For convenience, let $H := G - (V(D) - \{r\})$, let $P := D - r$, and let a, b be the ends of P. If H is 2-connected, then $C := D$ gives the desired cycle, and in this case, $m = 0$ and no P_t may exist. So assume that H is not 2-connected. Let $X := \{v_1, v_2, \dots, v_n\}$ be the set of cut vertices of H which are contained in B_D. Observe that $r \notin X$. Let $B_i^1, B_i^2, \dots, B_i^{n_i}$ denote the v_i-bridges of H other than B_D, where $n_i \geq 1$ because v_i is a cut vertex of H. Let $\mathcal{B} := \{B_i^j \ : \ 1 \leq i \leq n, 1 \leq j \leq n_i\}$. Note that $r \notin V(B_i^j) \cup N(B_i^j)$ for any $B_i^j \in \mathcal{B}$.

Because G is 4-connected, $B_i^j - v_i$ has at least three neighbors on P. Let a_i^j, b_i^j be the neighbors of $B_i^j - v_i$ on P such that $a_i^j P b_i^j$ is maximal and a, a_i^j, b_i^j, b occur on P in this order. See Figure 3. For convenience, let $P_i^j := a_i^j P b_i^j$, and let $Q_i^j = P_i^j - \{a_i^j, b_i^j\}$. We have the following two observations.

(b) $V(Q_i^j) \neq \emptyset$ and $N(Q_i^j) \cap V(B_i^j - v_i) \neq \emptyset$.

(c) Because G is 4-connected, $N(Q_i^j) \not\subset V(B_i^j) \cup V(D)$.

Claim 1. *For each B_i^j, there exists a $D_i^j \in \mathcal{D}$ such that*

(i) $V(D_i^j) \cap (V(H) - V(B_i^j)) = \{r\}$,

(ii) $v_i \notin V(D_i^j)$, *and*

(iii) $V(D_i^j) \cap V(Q_i^j) = \emptyset$.

Proof of Claim: Consider the graph $G_i^j := G\left[V(B_i^j) \cup \{a_i^j, b_i^j\}\right]$. Let $S = \{v_i, a_i^j, b_i^j\} \cup (N(Q_i^j) \cap V(B_i^j))$. Since G is 4-connected, for any $T \subset V(G_i^j)$ with $|T| \leq 3$, every component of $G - T$ must contain an element of S. Further, since B_i^j is a v_i-bridge of H, there must exist an $a_i^j - b_i^j$ path in $G_i^j - v_i$. Applying Lemma 2.5 (with G_i^j, a_i^j, b_i^j, v_i as $G, a, a', b = b'$, respectively), there must exist an induced $a_i^j - b_i^j$ path S_i^j in $G_i^j - v_i$ such that if F is a component of $G_i^j - V(S_i^j)$, then F contains some element of S.

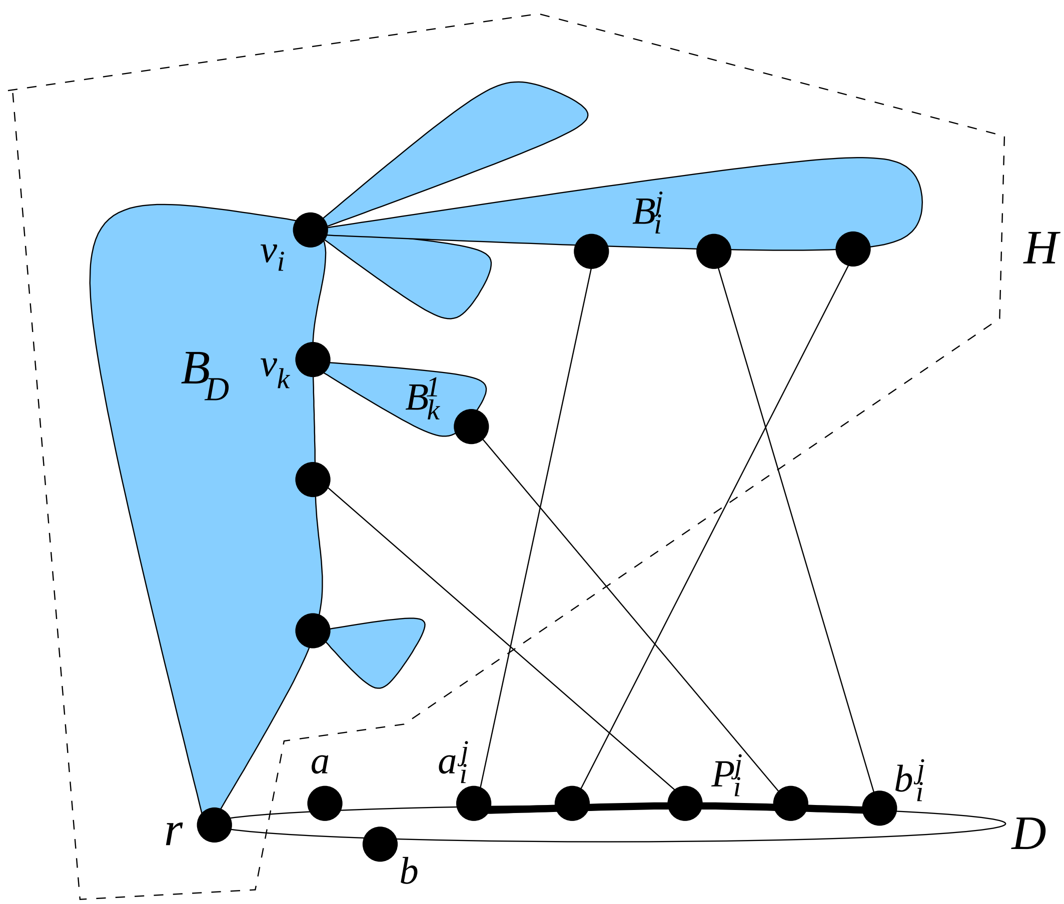

Figure 2: Theorem 3.1

Let $D_i^j := (D - V(Q_i^j)) \cup S_i^j$. Then D_i^j is a cycle in G. By construction, (i) $V(D_i^j) \cap (V(H) - V(B_i^j)) = \{r\}$, (ii) $v_i \notin V(D_i^j)$, and (iii) $V(D_i^j) \cap V(Q_i^j) = \emptyset$. Note that $e \in E(D_i^j)$. It remains to show that $D_i^j \in \mathcal{D}$.

Because D and S_i^j are induced subgraphs of G and by the definition of a_i^j and b_i^j, D_i^j is an induced cycle in G. So we need to show that $G - (V(D_i^j) - \{r\})$ is connected, and r is not a cut vertex of $G - \Big(V(D_i^j) - \{r\}\Big)$. Since $V(B_D \cap D_i^j) = V(B_D \cap D) = \{r\}$, then $B_D - r \subseteq G - V(D_i^j)$. Since $B_D - r$ is connected and D_i^j is induced, it suffices to show that for each $x \in V(G) - V(D_i^j)$, $G - V(D_i^j)$ has a path from x to $V(B_D) - \{r\}$.

Suppose $x \in V(B_k^l)$ for some $B_k^l \neq B_i^j$. By construction, $V(B_k^l) \cap V(D_i^j) = \emptyset$. Thus, B_k^l (and hence $G - V(D_i^j)$) has a path from x to $v_k \in V(B_D) - \{r\}$. So assume $x \in V(B_i^j) \cup V(Q_i^j)$.

If $x \in V(Q_i^j)$ then, since $N(Q_i^j) \not\subseteq V(B_i^j) \cup V(D)$ (by **(c)**) and $V(D_i^j) \cap V(Q_i^j) = \emptyset$ (by (iii)), $G - V(D_i^j)$ has a path from x to $V(B_D) - \{r\}$.

So let $x \in V(B_i^j)$. Let F denote the component of $G_i^j - V(S_i^j)$ containing x. If $v_i \in V(F)$, then F (and hence $G - V(D_i^j)$) contains a path from x to $v_i \in V(B_D) - \{r\}$. So assume that F has a neighbor of Q_i^j. Since $N(Q_i^j) \not\subseteq V(B_i^j) \cup V(D)$ (again, by **(c)**) and $V(D_i^j) \cap V(Q_i^j) = \emptyset$ (again, by (iii)), then $G - V(D_i^j)$ must have a path from x to $V(B_D) - \{r\}$. □

For each $x \in V(H)$, we define x^* as follows. If $x \in V(B_i^j)$ for some i, j, then let $x^* = v_i$. If $x \in V(B_D)$, then define $x^* = x$.

Claim 2. *For any $B_i^j \in \mathcal{B}$ and for any $x, y \in \Big(N(Q_i^j) \cap V(H)\Big) - V(B_i^j)$, $x^* = y^*$.*

Proof of Claim: Suppose that there are $x, y \in \Big(N(Q_i^j) \cap V(H)\Big) - V(B_i^j)$ such that $x^* \neq y^*$. Then G contains disjoint paths X and Y joining x to x^* and y to y^* respectively such that both X and Y are also disjoint from $D \cup (B_i^j - v_i) \cup (B_D - \{x^*, y^*\})$. Let $x', y' \in V(Q_i^j)$ such that $xx', yy' \in E(G)$. By Claim 1, there is some $D_i^j \in \mathcal{D}$ such that $V(D_i^j) \cap (V(H) - V(B_i^j)) = \{r\}$, $v_i \notin V(D_i^j)$, and $V(D_i^j) \cap V(Q_i^j) = \emptyset$. Then both B_D and the $x^* - y^*$ path $X \cup xx'Q_i^jy'y \cup Y$ are contained in $B_{D_i^j}$. Hence $\left|V\left(B_{D_i^j}\right)\right| > |V(B_D)|$, and so D_i^j contradicts **(a)**. □

Define a new graph $\mathcal{K}$ such that $V(\mathcal{K}) = \mathcal{B}$, and $B_i^j B_k^l \in E(\mathcal{K})$ if and only if $E(P_i^j) \cap E(P_k^l) \neq \emptyset$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be the components of $\mathcal{K}$. For each $t \in \{1, \ldots, m\}$, let $V_t := \{v_i : B_i^j \in V(\mathcal{A}_t)$ for some $1 \leq j \leq n_i\}$, $P'_t := \bigcup_{B_i^j \in V(\mathcal{A}_t)} P_i^j$, and $B_t := \bigcup_{B_i^j \in V(\mathcal{A}_t)} B_i^j$. By definition, each P'_t is a subpath of P, $E(P'_s) \cap E(P'_t) = \emptyset$ for all $s \neq t$. Without loss of generality, assume that $P'_1, \ldots, P'_m$ occur on P from a to b in the order listed. Let a_t and b_t be the ends of P'_t such that

a, a_t, b_t, b occur on P in this order, and let $Q_t := P'_t - \{a_t, b_t\}$. See Figure 3 for an example with $t = 3$.

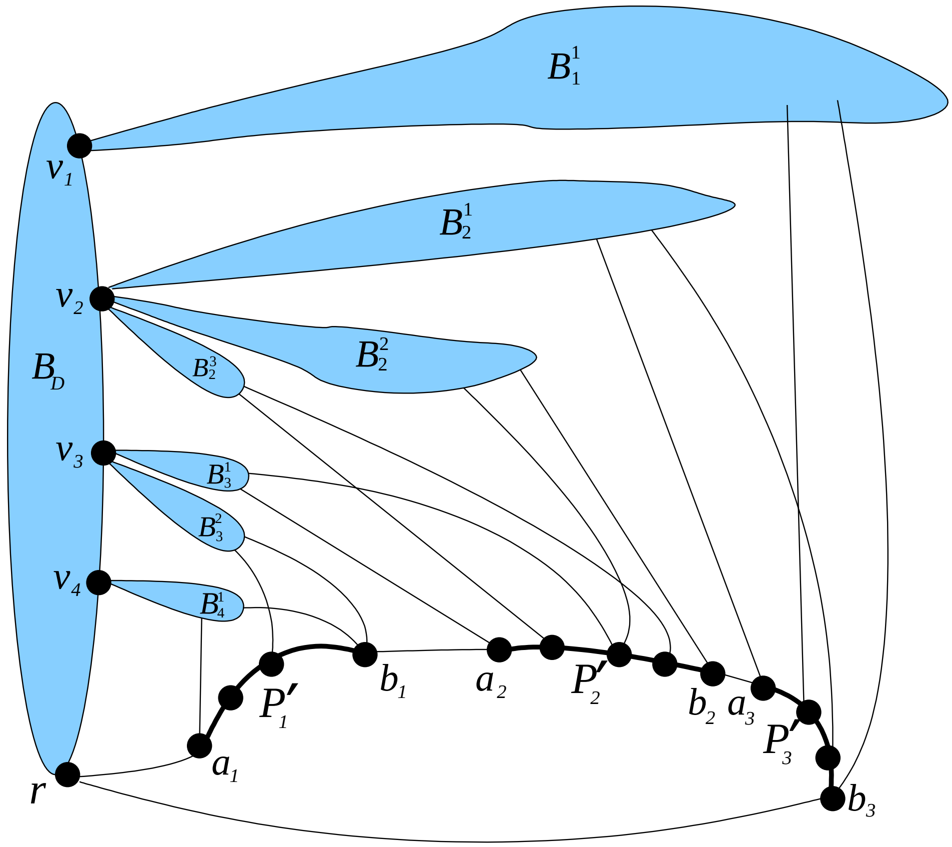

Figure 3: Claims 2 and 3

Claim 3. *For each* $t \in \{1, \ldots, m\}$, $|V_t| \leq 2$.

Proof of Claim: Assume that $|V_t| \geq 3$.

Case (1). $\mathcal{A}_t$ contains an induced path $B_i^j B_k^l B_p^q$ with $i \neq p$.

Then $E(P_i^j) \cap E(P_p^q) = \emptyset$, $E(P_i^j) \cap E(P_k^l) \neq \emptyset$, and $E(P_k^l) \cap E(P_p^q) \neq \emptyset$. Hence, we may assume that the vertices a, a_i^j, b_i^j, a_p^q, b_p^q, b occur on P in the order listed, and that the vertices a, a_k^l, b_i^j, a_p^q, b_k^l, b occur on P in the order listed. Moreover, $a_k^l \neq b_i^j$ and $a_p^q \neq b_k^l$. Let $x \in V(B_i^j) - \{v_i\}$ such that $xb_i^j \in E(G)$, and let $y \in V(B_p^q) - \{v_p\}$ such that $ya_p^q \in E(G)$. Then $x, y \in \big(N(Q_k^l) \cap V(H)\big) - V(B_k^l)$ and $x^* = v_i \neq v_p = y^*$, contradicting Claim 2.

Case (2). $\mathcal{A}_t$ contains a triangle $B_i^j B_k^l B_p^q B_i^j$ with $i \neq k \neq p \neq i$.

First, we prove that one of the following must be true: $N(Q_i^j) \cap (V(B_k^l) - \{v_k\}) \neq \emptyset$ and $N(Q_i^j) \cap (V(B_p^q) - \{v_p\}) \neq \emptyset$, or $N(Q_k^l) \cap (V(B_i^j) - \{v_i\}) \neq \emptyset$ and $N(Q_k^l) \cap (V(B_p^q) - \{v_p\}) \neq \emptyset$, or $N(Q_p^q) \cap (V(B_i^j) - \{v_i\}) \neq \emptyset$ and $N(Q_p^q) \cap (V(B_k^l) - \{v_k\}) \neq \emptyset$. Assume from symmetry that a, a_i^j, a_k^l, a_p^q, b, not necessarily distinct, occur on P in that order. Because $E(P_i^j) \cap E(P_p^q) \neq \emptyset$, then $b_i^j \in V(a_p^q P b - a_p^q)$. Similarly, $b_k^l \in V(a_p^q P b - a_p^q)$. If $a_i^j \neq a_k^l$, then $\emptyset \neq N(a_k^l) \cap (V(B_k^l) - \{v_k\}) \subseteq N(Q_i^j) \cap (V(B_k^l) - \{v_k\})$ and $\emptyset \neq N(a_p^q) \cap (V(B_p^q) - \{v_p\}) \subseteq N(Q_i^j) \cap (V(B_p^q) - \{v_p\})$. So assume $a_i^j = a_k^l$. Then from symmetry we may assume that $b_i^j \in V(a_p^q P b_k^l)$. Hence $Q_i^j \subset Q_k^l$. If $a_k^l \neq a_p^q$, then from **(b)**, $\emptyset \neq N(Q_i^j) \cap (V(B_i^j) - \{v_i\}) \subseteq N(Q_k^l) \cap (V(B_i^j) - \{v_i\})$, and $\emptyset \neq N(a_p^q) \cap (V(B_p^q) - \{v_p\}) \subseteq N(Q_k^l) \cap (V(B_p^q) - \{v_p\})$. So assume $a_k^l = a_p^q$. We may now assume from symmetry that $a, a_i^j, b_i^j, b_k^l, b_p^q, b$, not necessarily distinct, occur on P in that order. Then $Q_i^j \subset Q_k^l \subset Q_p^q$, and from **(b)**, $\emptyset \neq N(Q_i^j) \cap (V(B_i^j) - \{v_i\}) \subseteq N(Q_p^q) \cap (V(B_i^j) - \{v_i\})$ and $\emptyset \neq N(Q_k^l) \cap (V(B_k^l) - \{v_k\}) \subseteq N(Q_p^q) \cap (V(B_k^l) - \{v_k\})$.

By symmetry, we may assume $N(Q_k^l) \cap (V(B_i^j) - \{v_i\}) \neq \emptyset$ and $N(Q_k^l) \cap (V(B_p^q) - \{v_p\}) \neq \emptyset$. Then there exist $x \in N(Q_k^l) \cap (V(B_i^j) - \{v_i\})$ and $y \in N(Q_k^l) \cap (V(B_p^q) - \{v_p\})$. Hence $x, y \in \big(N(Q_k^l) \cap V(H)\big) - V(B_k^l)$ and $x^* = v_i \neq v_p = y^*$, contradicting Claim 2.

Case (3). Neither Case (1) nor Case (2).

Because Case (1) does not occur, for any induced path $B_i^j B_k^l B_p^q$ in $\mathcal{A}_t$, we must have $i = p$. Because Case (2) does not occur and since $|V_t| \geq 3$, $\mathcal{A}_t$ is not complete. Further, for any induced path R in $\mathcal{A}_t$, R contains no subpath $B_i^j B_k^l B_p^q$ with $i \neq p$. Hence R may only take on two forms: (I) $V(R)$ may be composed of B_i^j's for a fixed i, or (II) R may be alternating between B_i^j's and B_k^l's, for fixed i, k. Since we assume $|V_t| \geq 3$, then $\mathcal{A}_t$ contains B_i^j, B_k^l, B_p^q such that $i \neq k \neq p \neq i$. Choose an induced path R_1 in $\mathcal{A}_t$ from B_i^j to B_k^l, and another induced path R_2 in $\mathcal{A}_t$ from B_k^l to B_p^q. Clearly these paths cannot be of type (I), and so must be of type (II). But then $R_1 \cup R_2$ contains a subpath $B_i^{j_0} B_k^l B_p^{q_0}$ such that $i \neq k \neq p \neq i$, and we would have Case (1) or Case (2), a contradiction. $\square$

Claim 4. *For each* $t \in \{1, \ldots, m\}$ *such that* $|V_t| = 1$, *there exists* $d_t \in V(B_D) - (V_t \cup \{r\})$ *such that* $N(Q_t) - (V(B_t) \cup V(D)) = \{d_t\}$.

Proof of Claim: Suppose $|V_t| = 1$. Because G is 4-connected and D is induced in G, Q_t must have a neighbor $x \in V(B_D) - (V_t \cup \{r\})$. By the definition of Q_t, we may assume that $x \in N(Q_i^j)$, and we choose such Q_i^j to be maximal. If $N(Q_t) - (V(B_t) \cup V(D)) = \{x\}$, then $d_t := x$ is the desired vertex. So we assume that there is some $y \in N(Q_t) - (V(B_t) \cup V(D))$ such that $y \neq x$. Then $y \in V(B_D) - (V_t \cup \{r, x\})$. Because $x, y \in V(B_D)$, $x^* = x$ and $y^* = y$. By Claim 2 and because $x^* = x \neq y = y^*$, $y \notin N(Q_i^j)$. Hence, $|\mathcal{A}_t| \geq 2$, and so, there exists some $B_k^l \in V(\mathcal{A}_t) - \{B_i^j\}$ such that $E(P_k^l) \cap E(P_i^j) \neq \emptyset$. By the maximality of Q_i^j, Q_i^j is not a proper subpath of Q_k^l, so either $a_k^l \in V(Q_i^j)$ or $b_k^l \in V(Q_i^j)$ or $Q_k^l = Q_i^j$. By **(b)**, $N(Q_k^l) \cap (V(B_k^l) - \{v_k\}) \neq \emptyset$. Hence, Q_i^j has a neighbor $z \in V(B_k^l) - \{v_k\}$. Then $x, z \in \Big(N(Q_i^j) \cap V(H)\Big) - V(B_i^j)$ and $z^* = v_k \neq x^* = x$. But $v_k \in V_t$ and $x \notin V_t$; this contradicts Claim 2. □

Claim 5. *For each $t \in \{1, \ldots, m\}$ such that $|V_t| = 2$, $N(Q_t) \cap V(B_D) \subset V_t$.*

Proof of Claim: Suppose $|V_t| = 2$, and assume that there is some $x \in (N(Q_t) \cap V(B_D)) - V_t$. Then $x^* = x \notin V_t$. By definition of Q_t, $x \in V(Q_i^j)$ for some $Q_i^j \in V(\mathcal{A}_t)$, and we choose such Q_i^j to be maximal. Because $|V_t| \geq 2$, $|\mathcal{A}_t| \geq 2$. Hence there exists some $B_k^l \in V(\mathcal{A}_t) - \{B_i^j\}$ such that $E(P_k^l) \cap E(P_i^j) \neq \emptyset$. By the maximality of Q_i^j, Q_i^j is not a proper subpath of Q_k^l, so either $a_k^l \in V(Q_i^j)$ or $b_k^l \in V(Q_i^j)$ or $Q_k^l = Q_i^j$. From **(b)**, $N(Q_k^l) \cap (V(B_k^l) - \{v_k\}) \neq \emptyset$. Hence Q_i^j has a neighbor $y \in V(B_k^l) - \{v_k\}$. Note that $y^* = v_k \in V_t$ and $x^* = x \notin V_t$. Hence $x^* \neq y^*$. But $x, y \in \Big(N(Q_i^j) \cap V(H)\Big) - V(B_i^j)$, contradicting Claim 2. □

From Claims 3, 4, and 5, we may now identify the paths $P_1, \ldots, P_m$ and vertices a_t, b_t, c_t, d_t, $1 \leq t \leq m$, given in the statement of Theorem 3.1. We will then verify conditions (i) and (ii) in the conclusion of this theorem.

If $|V_t| = 2$, then let $V_t := \{c_t, d_t\}$, and let $G_t := G[V(B_t) \cup V(P_t')]$. If $|V_t| = 1$, then by Claim 4, $N(Q_t) - (V(B_t) \cup V(D)) = \{d_t\} \subset V(B_D)$, and so, let $V_t := \{c_t\}$ and $G_t := G\left[V(B_t) \cup \{d_t\} \cup V(P_t')\right]$. From Claims 4 and 5, $G_t - \{a_t, b_t, c_t, d_t\}$ is a component of $G - \{a_t, b_t, c_t, d_t\}$. We proceed to prove (i) and (ii). To do so, we will replace P_t' with an $a_t - b_t$ Hamilton path P_t in $G_t - \{c_t, d_t\}$. First, we establish the following fact.

Claim 6. *The ordered quintuple $(G_t, a_t, c_t, b_t, d_t)$ is planar.*

Proof of Claim: Since G is 4-connected, if $T \subset V(G_t)$ with $|T| \leq 3$, then any component of $G_t - T$ must contain an element of $\{a_t,\ b_t,\ c_t,\ d_t\}$. We may apply Corollary 2.2 to G_t, a_t, b_t, c_t, d_t (as G, u_1, v_1, u_2, v_2 respectively). Then either (1) G_t has disjoint paths joining a_t to b_t and c_t to d_t respectively or (2) $(G_t, a_t, c_t, b_t, d_t)$ is planar. If (2) holds, then we have our claim. So assume that (1) holds.

We may apply Lemma 2.5 to G_t, a_t, b_t, c_t, d_t (as G, a, a', b, b' respectively), letting $S = \{a_t, b_t, c_t, d_t\}$, and find an induced $a_t - b_t$ path R in $G_t - \{c_t, d_t\}$ such that every component of $G_t - V(R)$ contains an element of S. Let $D' := (D - V(Q_t)) \cup R$. Then D' is an induced cycle in G and $G - V(D')$ is connected. It is then easy to see that $D' \in \mathcal{D}$. But both B_D and a $c_t - d_t$ path in $G_t - V(R)$ are contained in $B_{D'}$. Thus $|V(B_{D'})| > |V(B_D)|$, contradicting **(a)**. $\square$

Since G is 4-connected, if $T \subset V(G_t)$ with $|T| \leq 3$, then every component of $G_t - T$ must contain an element of $\{a_t,\ b_t,\ c_t,\ d_t\}$. We may now apply Corollary 2.4 (with $(G_t, a_t, c_t, b_t, d_t)$ as (G, a, c, b, d)) to create an $a_t - b_t$ Hamilton path P_t in $G_t - \{c_t, d_t\}$, for each $t \in \{1, \ldots, m\}$. By construction, $P_1, \ldots, P_m$ are all edge-disjoint paths. We let $C := (D - (\bigcup_{t=1}^{m} V(Q_t))) \cup (\bigcup_{t=1}^{m} P_t)$. Then C is a cycle in G and $e \in E(C)$, and $G - (V(C) - \{r\}) = B_D$ is 2-connected. Note that $G_t = G[V(P_t) \cup \{c_t, d_t\}]$, and $G_t - \{a_t, b_t, c_t, d_t\} = G[V(P_t) - \{a_t, b_t\}]$ is a component of $G - \{a_t, b_t, c_t, d_t\}$. Hence $P_t,\ a_t,\ b_t,\ c_t,\ d_t, 1 \leq t \leq m$, satisfy condition (ii). We may easily see that condition (i) is also satisfied. Suppose there is a chord xy of C with $\{x, y\} \not\subset V(P_t)$ for all $1 \leq t \leq m$. If $x, y \notin V(Q_t)$ for any t, then xy is a chord of D. But D is induced in G, and this is a contradiction. So assume that $y \in V(Q_t)$ for some t, and then $x \notin V(P_t)$, contradicting the fact that $G_t - \{a_t, b_t, c_t, d_t\}$ is a component of $G - \{a_t, b_t, c_t, d_t\}$.

This completes the proof of Theorem 3.1. $\square$

As a corollary, we have Theorem 1.2 by setting $e = ra$.

4 5-Connected graphs and planar graphs

In the proof of Theorem 3.1, we choose a cycle D to maximize a block B_D of $H = G - (V(D) - \{r\})$. After a sequence of five Claims, we showed that any v_i-bridge other than B_D in H could be enclosed within a subgraph associated with a 4-cut. In a 5-connected graph, these 4-cuts cannot exist; this is the inspiration for Theorem 1.3. However, since we are now interested in the connectivity of $G - V(C)$, we must ensure that a non-trivial block (i.e., one with at least three vertices) exists in $G - V(C)$.

Since the proof of Theorem 1.3 closely parallels the proof of Theorem 3.1, we give only an outline and refer the reader to Section 3, where possible. Following the proof, we demonstrate the relation of Theorem 1.3 to Lovász's conjecture.

Proof of Theorem 1.3. Let G be a 5-connected graph, and let $e = ab$.

Claim 0. *There exists an induced cycle D through e in G such that $G - V(D)$ contains a non-trivial block.*

Proof of Claim. By Theorem 1.1, there exists an induced cycle F through e in G such that $G - V(F)$ is connected. If $G - V(F)$ contains a non-trivial block, then $D := F$ gives the desired cycle for the claim. So assume $G - V(F)$ does not contain a non-trivial block; then $G - V(F)$ is a tree. Let x be any leaf of $G - V(F)$, let y be the neighbor of x in $G - V(F)$, and let $J := G - (V(F) \cup \{x\})$. Since G is 5-connected, $|N(x) \cap V(F)| \geq 4$. Hence, $|N(J) \cap V(F)| \geq 4$; otherwise, $(N(J) \cap V(F)) \cup \{x\}$ would be a cut set of size ≤ 4 in G. Let $P := F - e$, and let a', b' be the neighbors of J on F such that $a'Pb'$ is maximal and a, a', b', b occur on P in this order. Let $P' := a'Pb'$. Note that $V(P') - \{a', b'\} \neq \emptyset$, since $|N(J) \cap V(F)| \geq 4$.

Let $S := (V(J) \cap N(P' - \{a', b'\})) \cup \{a', b', y\}$. Observe that for any set $T \subset V(J) \cup \{a', b'\}$ with $|T| \leq 3$, any component of $G[V(J) \cup \{a', b'\}] - T$ must contain an element of S. Hence, from Lemma 2.5 (with $G[V(J) \cup \{a', b'\}], a', b', y$ as $G, a, a', b = b'$ respectively), there exists an induced $a' - b'$ path R in $G[V(J) \cup \{a', b'\}]$ such that every component of $G[V(J) \cup \{a', b'\}] - V(R)$ contains an element of S. Note that the cycle $D := (F - (V(P') - \{a', b'\})) \cup R$ is induced. Note too that if $N(x) \cap (V(P') - \{a', b'\}) \neq \emptyset$, then $G - V(D)$ is connected, because every component of $G\,[V(J) \cup \{a', b'\}] - R$ contains y or some vertex of $N(P' - \{a', b'\}) \cap V(J)$.

Case (1) If $|N(x) \cap V(P' - \{a', b'\})| \geq 2$, then let c, d be distinct elements of $N(x) \cap V(P' - \{a', b'\})$. Then $G[\{x\} \cup V(cP'd)]$ contains a cycle. Since $G[\{x\} \cup V(cP'd)] \subseteq G - V(D)$, and $G - V(D)$ is connected, then $G - V(D)$ contains a cycle (and hence a nontrivial block). Therefore, D gives the desired cycle for the Claim.

Case (2) If $|N(x) \cap V(P' - \{a', b'\})| \leq 1$, then $|N(x) \cap V(P - (V(P') - \{a', b'\}))| \geq 3$. Hence, $V(F) - V(P') \neq \emptyset$. Because $N(J) \subseteq V(P')$ and F is induced in G, every vertex in $V(F) - V(P')$ has degree at most 3. This contradicts that G is 5-connected.

Hence, $G - V(D)$ contains a non-trivial block. $\square$

Let $\mathcal{D}$ denote the set of those induced cycles D in G such that $e \in E(D)$, and $G - V(D)$ contains a non-trivial block. For any $D \in \mathcal{D}$, let B_D denote a block of $G - V(D)$ such that $|V(B_D)|$ is maximum.

(a) We choose $D \in \mathcal{D}$ so that $|V(B_D)|$ is maximum.

For convenience, let $H := G - V(D)$ and $P := D - e$. If H is 2-connected, then $C := D$ is the desired cycle. So assume that H is not 2-connected. Let $X := \{v_1, v_2, \ldots, v_n\}$ be the set of cut vertices of H which are contained in B_D. Let $B_i^1, B_i^2, \ldots, B_i^{n_i}$ denote the v_i-bridges of H other than B_D, where $n_i \geq 1$ because v_i is a cut vertex of H. We let $\mathcal{B} := \{B_i^j \;:\; 1 \leq i \leq n, 1 \leq j \leq n_i\}$.

Because G is 5-connected, $B_i^j - v_i$ has at least four neighbors on P. Let a_i^j, b_i^j be the neighbors of $B_i^j - v_i$ on P such that $a_i^j P b_i^j$ is maximal and a, a_i^j, b_i^j, b occur on P in this order. As in the proof of Theorem 3.1, we let $P_i^j := a_i^j P b_i^j$, $Q_i^j := P_i^j - \{a_i^j, b_i^j\}$. We have the following two observations.

(b) $V(Q_i^j) \neq \emptyset$ and $N(Q_i^j) \cap V(B_i^j - v_i) \neq \emptyset$.

(c) $N(Q_i^j) \not\subset V(B_i^j) \cup V(D)$.

Claim 1. *For any B_i^j, there exists a $D_i^j \in \mathcal{D}$ such that (i) $V(D_i^j) \cap (V(H) - V(B_i^j)) = \emptyset$, (ii) $v_i \notin V(D_i^j)$, and (iii) $V(D_i^j) \cap V(Q_i^j) = \emptyset$.*

Proof of Claim. Showing such a cycle exists is nearly identical to the proof of Claim 1 in Section 3. Apply Lemma 2.5 (with $G[V(P_i^j) \cup V(B_i^j)], a_i^j, b_i^j$, and v_i as G, a, a', and $b = b'$, respectively) to create the cycle D_i^j through e. The only difference is that $V(D_i^j) \cap (V(H) - V(B_i^j)) = \emptyset$ in conclusion (i), since $V(H) \cap V(D) = \emptyset$. □

For any $x \in V(H)$, we may define x^* as in Section 3. Similarly, we define the auxiliary graph $\mathcal{K}$, its components $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$, the sets $V_1, V_2, \ldots, V_m$, the subgraphs $B_1, B_2, \ldots, B_m$, and the paths $P_1', P_2', \ldots, P_m', Q_1, Q_2, \ldots, Q_m$ as in Section 3. With the same proofs for Claims 2, 3, 4, and 5 in the proof of Theorem 3.1, appealing to Claim 1 above where necessary, we have the following claims.

Claim 2. *For each $B_i^j \in \mathcal{B}$ and for any $x, y \in \Big(N(Q_i^j) \cap V(H)\Big) - V(B_i^j)$, $x^* = y^*$.*

Claim 3. *For each $t \in \{1, \ldots, m\}$, $|V_t| \leq 2$.*

Claim 4. *For each $t \in \{1, \ldots, m\}$ such that $|V_t| = 1$, there exists $d_t \in V(B_D) - (V_t \cup \{r\})$ such that $N(Q_t) - (V(B_t) \cup V(D)) = \{d_t\}$.*

Claim 5. *For each $t \in \{1, \ldots, m\}$ such that $|V_t| = 2$, $N(Q_t) \cap V(B_D) \subset V_t$.*

If $|V_t| = 2$, then let $V_t := \{c_t, d_t\}$, and let $G_t := G[V(B_t) \cup V(P_t')]$. If $|V_t| = 1$, then by Claim 4, $N(Q_t) - (V(B_t) \cup V(D)) = \{d_t\} \subset V(B_D)$, and so, let $V_t := \{c_t\}$ and $G_t := G\,[V(B_t) \cup \{d_t\} \cup V(P_t')]$. From Claims 4 and 5 above, $G_t - \{a_t, b_t, c_t, d_t\}$ is a component of $G - \{a_t, b_t, c_t, d_t\}$. This is a contradiction, since G is 5-connected.

Hence H is 2-connected, completing our proof. □

As a consequence of Theorem 1.3, we derive the following result of [6] and [2].

Corollary 4.1. *Let G be a 5-connected graph and $x, y \in V(G)$ be distinct. Then G contains an induced $x - y$ path P such that $G - V(P)$ is 2-connected.*

Proof. If $xy \in E(G)$, then let P be the $x - y$ path with $E(P) = \{xy\}$. Since G is 5-connected, $G - V(P) = G - \{x, y\}$ is 2-connected. So assume that $xy \notin E(G)$. Let $G' := G + xy$ and let $e = xy$. Note that G' is 5-connected. By Theorem 1.3, G' has an induced cycle C through e such that $G' - V(C)$ is 2-connected. Let $P := C - e$. Then P is an induced path in G. Since $G - V(P) = G' - V(C)$, then $G - V(P)$ is 2-connected. □

Corollary 4.1 shows that if $f(k)$ (of Lovász's conjecture, mentioned in Section 1) exists, then $f(2) \leq 5$. The following example shows equality. Let G be the graph obtained from a cycle C on four vertices by adding two vertices x and y along with edges xa and ya for all $a \in V(C)$. Then G is 4-connected, but deleting any $x - y$ path leaves only a path.

Proof of Theorem 1.4 . Let G be a 4-connected planar graph, let C be a non-separating induced cycle in G, and let $r \in V(C)$. Since G is 4-connected, r must have at least four neighbors, and since C is induced, exactly two of those neighbors lie on C. Thus, $G-(V(C)-\{r\})$ is connected. Further, since $G - V(C)$ is connected, r is not a cut vertex of $G - (V(C) - \{r\})$.

Let B denote the block of $G - (V(C) - \{r\})$ containing r. Clearly, B is 2-connected. For convenience, let $P := C - r$, and let $H := G - V(P)$. Suppose that H is not 2-connected. Let $v \in V(B)$ such that v is a cut vertex of H (and hence, $v \neq r$), and let B' be a v-bridge of H such that $B' \neq B$. Let $x, y \in V(P) \cap N(B' - v)$ such that xPy is maximal. Since G is 4-connected, $G - \{x, y, v\}$ is connected; hence, $G - \{x, y, v\}$ has a path P' from $V(B' \cup xPy) - \{x, y, v\}$ to $V(B) - \{v\}$. Because C is an induced cycle in G, B' is a v-bridge of H, and r is not a cut vertex of H, then P' is a path from $V(xPy) - \{x, y\}$ to some $w \in V(B) - \{v, r\}$ which is also disjoint from $(V(B) - \{w\}) \cup V(B') \cup (V(C) - (V(xPy) - \{x, y\}))$. Let z be the end of P' in $V(xPy) - \{x, y\}$. See Figure 4.

Since B is 2-connected, there exist a $v - r$ path R_1 and $v - w$ path P'' in B such that $V(R_1 \cap P'') = \{v\}$. Let $P_1 := P'' \cup P'$, $P_2 := zPx$, $P_3 := zPy$, let R_3 be the subpath of $C - x$ between y and r, and let R_2 be the subpath of $C - y$ between x and r.

Let $x', y' \in V(B') - \{v\}$ such that $xx', yy' \in E(G)$. Since $B' - v$ is connected, $B' - v$ contains a path Q from x' to y'. Note that B' contains a path Q_1 from v to some $s \in V(Q)$ such that

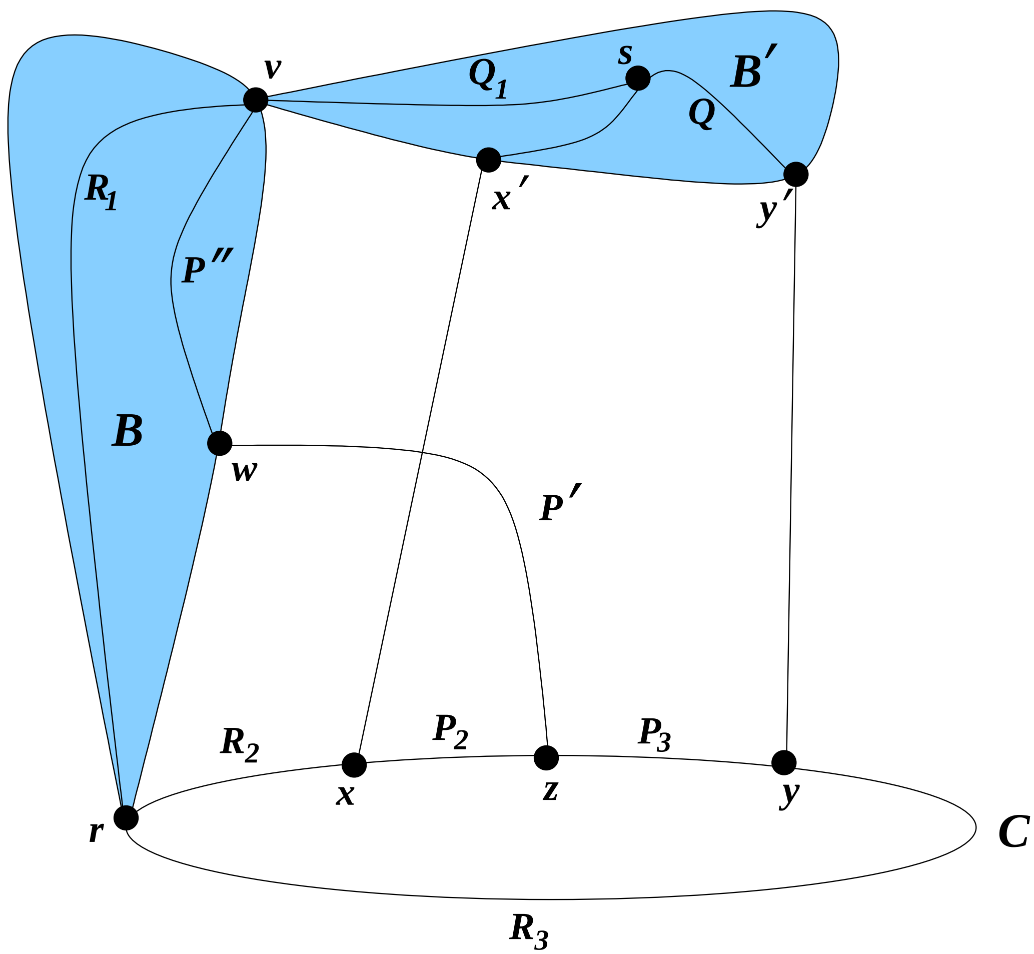

Figure 4: Theorem 1.4

$V(Q_1 \cap Q) = \{s\}$. Let $Q_2 := sQx'x$ and $Q_3 := sQy'y$.

Then $\left(\bigcup_{i=1}^{3} P_i\right) \cup \left(\bigcup_{i=1}^{3} Q_i\right) \cup \left(\bigcup_{i=1}^{3} R_i\right)$ is a subdivision of $K_{3,3}$. Hence G is not planar, contradicting our hypothesis. $\square$

5 Concluding remarks

Theorem 1.3 suggests that Theorem 1.1 might be generalized.

Conjecture. *For any positive integer k, there exists some positive integer $f(k)$ such that if graph G is $f(k)$-connected, then for any $e \in E(G)$, there exists an induced cycle C through e in G such that $G - V(C)$ is k-connected.*

This would imply Lovász's conjecture in exactly the same way that Theorem 1.3 implies the case for $k = 2$. Our proof for $k = 2$ relied on the highly useful block decomposition of connected graphs. Therefore, a natural problem is how to generalize block decomposition to k-connected graphs.

Our short-term goal is to find a non-separating ear decomposition for 4-connected graphs which will yield four independent spanning trees rooted at a vertex. Theorem 3.1 provides the first ear. It is not incidental that the cycle in Theorem 3.1 has planar sections. Huck in [4] proved the existence of four independent trees in every 4-connected planar graph. We intend to produce four independent trees in any 4-connected graph by building an ear decomposition with numerous planar sections and applying Huck's result. We hope that this will lend insight to an approach which could work for higher connectivity.

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