Disjoint Paths in Graphs III, Characterization

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Abstract

Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$ such that $\{a,b,c\} \neq \{a',b',c'\}$. We say that $(G,\{a,c\},\{a',c'\},(b,b'))$ is an *obstruction* if, for any three vertex disjoint paths from $\{a,b,c\}$ to $\{a',b',c'\}$ in G, one path is from b to b'. In this paper we characterize obstructions.

*Partially supported by NSF grant DMS-9970527 and NSA grant MDA904-03-1-0052

1 Introduction

We use the terminology in [5]. Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$. Then $(G, \{a,c\}, \{a',c'\}, (b,b'))$ is an *obstruction* if for any three vertex disjoint paths from $\{a,b,c\}$ to $\{a',b',c'\}$ in G, one path is from b to b'. In [6], a special class of obstructions are characterized. In this paper, we characterize all obstructions. In order to state our main result, we need the following definition from [5].

A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

(a) for $1 \leq i \neq j \leq k$, $N(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and

(c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc D with no pair of edges crossing such that, for each A_i with $|N(A_i)| = 3$, $N(A_i)$ induces a facial triangle in $p(G, \mathcal{A})$.

If, in addition, $b_0, b_1, \ldots, b_n$ are vertices of G such that $b_i \notin A_j$ for any $A_j \in \mathcal{A}$ and $b_0, b_1, \ldots, b_n$ occur on the boundary of D in that cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_0, b_1, \ldots, b_n)$ is 3-planar.

The building blocks of obstructions are described in the following definition.

(1.1) Definition. Let G be a graph, $\{a,b,c\} \subseteq V(G)$, and $\{a',b',c'\} \subseteq V(G)$. Suppose $\{a,b,c\} \neq \{a',b',c'\}$, and assume that G has no 3-separation (G_1, G_2) such that $\{a,b,c\} \subseteq G_1$ and $\{a',b',c'\} \subseteq G_2$. Then we call $(G, (a,b,c), (a',b',c'))$ a *rung* if one of the following conditions is satisfied:

(1) $b = b'$ or $\{a,c\} = \{a',c'\}$;

(2) $a = a'$ and $(G - a, c, c', b', b)$ is 3-planar;

(2′) $c = c'$ and $(G - c, a, a', b', b)$ is 3-planar;

(3) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$ and (G, a', b', c', c, b, a) is 3-planar;

(4) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a 1-separation (G_1, G_2) such that $\{a,a',b,b'\} \subseteq G_1$, $\{c,c'\} \subseteq G_2$, and (G_1, a, a', b', b) is 3-planar;

(4′) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, G has a 1-separation (G_1, G_2) such that $\{c,c',b,b'\} \subseteq G_1$, $\{a,a'\} \subseteq G_2$, and (G_1, c, c', b', b) is 3-planar;

(5) $\{a,b,c\} \cap \{a',b',c'\} = \emptyset$, (G, a, a', b', b) is 3-planar, and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z, b\}$ (or $V(G_1 \cap G_2) = \{z, b'\}$), $\{a,a',b,b'\} \subseteq G_1$, $\{c,c'\} \subseteq G_2$, and (G_2, c, c', z, b) (or (G_2, c, c', b', z)) is 3-planar;

(5′) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, (G,c,c',b',b) is 3-planar, and G has a separation (G_1,G_2) such that $V(G_1\cap G_2)=\{z,b\}$ (or $V(G_1\cap G_2)=\{z,b'\}$), $\{c,c',b,b'\}\subseteq G_1$, $\{a,a'\}\subseteq G_2$, and (G_2,a,a',z,b) (or (G_2,a,a',b',z)) is 3-planar;

(6) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{u,w\}$, $V(G_c\cap M)=\{p,q\}$, $V(G_a\cap G_c)=\emptyset$, $\{a,a',b'\}\subseteq G_a$, $\{c,c',b\}\subseteq G_c$, (G_a,a,a',b',w,u) is 3-planar, and (G_c,c',c,b,p,q) is 3-planar;

(6′) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{u,w\}$, $V(G_c\cap M)=\{p,q\}$, $V(G_a\cap G_c)=\emptyset$, $\{a,a',b\}\subseteq G_a$, $\{c,c',b'\}\subseteq G_c$, (G_a,b,a,a',w,u) is 3-planar, and (G_c,b',c',c,p,q) is 3-planar;

(7) $\{a,b,c\}\cap\{a',b',c'\}=\emptyset$, and there are pairwise edge disjoint subgraphs G_a,G_c,M of G such that $G=G_a\cup G_c\cup M$, $V(G_a\cap M)=\{b,b',w\}$, $V(G_c\cap M)=\{b,b',p\}$, $V(G_a\cap G_c)=\{b,b'\}$, $\{a,a'\}\subseteq G_a$, $\{c,c'\}\subseteq G_c$, (G_a,a,a',b',w,b) is 3-planar, and (G_c,c',c,b,p,b') is 3-planar.

(1.2) Definition. Let L be a graph and let $R_1,\ldots,R_m$ be edge disjoint subgraphs of L such that

(i) $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$ is a rung for each $1\le i\le m$,

(ii) $V(R_i\cap R_j)=\{x_i,v_i,y_i\}\cap\{x_{j-1},v_{j-1},y_{j-1}\}$ for $1\le i<j\le m$,

(iii) for any $1\le i<j\le m$, if $x_i=x_j$ then $x_k=x_i$ for all $i\le k\le j$, if $v_i=v_j$ then $v_k=v_i$ for all $i\le k\le j$, and if $y_i=y_j$ then $y_k=y_i$ for all $i\le k\le j$.

(iv) $L=(\bigcup_{i=1}^m R_i)+S$, where S consists of edges of L with both ends in some $\{x_i,v_i,y_i\}$, $1\le i\le m$.

Then we call $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ a *ladder with rungs* $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i))$, $i=1,\ldots,m$, or simply, a *ladder along* $v_0\ldots v_m$.

It is easy to see that a rung is at most 5-connected. It was shown in [5] (Proposition 4.4) that if $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ is a ladder, then $(L,\{x_0,y_0\},\{x_m,y_m\},(v_0,v_m))$ is an obstruction. The main result of this paper states that every obstruction can be constructed from ladders and 3-planar graphs in a special way. For a sequence S, the *reduced sequence* of S is the sequence obtained from S by removing all but one consecutive identical elements. For example, the reduced sequence of *aaabcca* is *abca*.

(1.3) Theorem. *Let G be a graph, $\{a,b,c\}\subseteq V(G)$, and $\{a',b',c'\}\subseteq V(G)$ such that $\{a,b,c\}\ne\{a',b',c'\}$. Assume that, for any $T\subseteq V(G)$ with $|T|\le 3$, every component of $G-T$ contains some element of $\{a,b,c\}\cup\{a',b',c'\}$. Then $(G,\{a,c\},\{a',c'\},(b,b'))$ is an obstruction iff one of the following statements holds.*

(1) *G has a separation (G_1, G_2) of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.*

(2) *$(G, (a, b, c), (a', b', c'))$ is a ladder.*

(3) *G has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$.*

Note the condition of (1.3) that for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains some element of $\{a, b, c\} \cup \{a', b', c'\}$. It is a natural condition, for the following reason. Suppose that $T \subseteq V(G)$, $|T| \leq 3$, and $G - T$ contains a component H with $V(H) \cap (\{a, b, c\} \cup \{a', b', c'\}) = \emptyset$. Let G' be obtained from G by removing H and adding new edges between every pair of distinct vertices in $N(H)$. Then it is easy to see that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction iff $(G', \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction.

Also note that if (1) of (1.3) holds, then there do not exist three disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$, and hence, $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction in a trivial sense.

As a consequence of (1.3), we will prove the following result.

(1.4) Corollary. *Let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be an obstruction. If $\{a, c\} \neq \{a', c'\}$ and $b \neq b'$, then G is at most 7-connected.*

The rest of the paper is organized as follows. In Section 2, we prove a technical lemma. In Section 3, we prove (1.3). In Section 4, we will prove (1.4) and construct 7-connected obstructions.

2 Good ladders

The main goal of this section is to prove a technical lemma, based on good ladders. To this end, we need the following result proved in [2] (also independently in [1] and [3]). Also, see (2.4) of [5].

(2.1) Theorem. *Let G be a graph and $\{s_1, s_2, t_1, t_2\} \subseteq V(G)$. Then G contains no $(\{s_1, t_1\}, \{s_2, t_2\})$-linkage iff (G, s_1, s_2, t_1, t_2) is 3-planar.*

We also need the following result, which follows easily from Proposition 3.2 of [5].

(2.2) Lemma. *Let $(G, \mathcal{A})$ be 3-planar and let $b, b' \in V(G)$ be distinct. Let $u, v \in V(G)$ such that $u, v \in p(G, \mathcal{A})$. If $p(G, \mathcal{A})$ contains a $(\{u, b\}, \{v, b'\})$-linkage, then G contains a $(\{u, b\}, \{v, b'\})$-linkage L.*

To prove the main result of this section, we need the main result of [6]. For convenience, we state the following definition.

(2.3) Definition. Let $(G,(a,b,c),(a',b',c'))$ be a rung. Suppose that $G-\{b,b'\}$ has disjoint paths A, C from a, c to a', c', respectively, and assume that the following conditions are satisfied:

(1) If G is connected and $b \neq b'$, then $G-((A-a')\cup(C-c))$ contains a $(\{b,a'\},\{c,b'\})$-linkage, and $G-((A-a)\cup(C-c'))$ contains an $(\{a,b'\},\{b,c'\})$-linkage;

(1′) if G is connected and $b = b'$, then $G-((A-a)\cup C)$ has a path from a to b', $G-((A-a')\cup C)$ has a path from a' to b, $G-(A\cup(C-c))$ has a path from c to b', and $G-(A\cup(C-c'))$ has a path from c' to b;

(2) if G is not connected and $\{a,b\}\cup\{a',b'\}$ is contained in a component of G, then $G-((A-a)\cup C)$ contains a path from a to b' (and not using b if $b\neq b'$) and $G-((A-a')\cup C)$ contains a path from b to a' (and not using b' if $b\neq b'$); and

(2′) if G is not connected and $\{b,c\}\cup\{b',c'\}$ is contained in a component of G, then $G-(A\cup(C-c))$ contains a path from c to b' (and not using b if $b\neq b'$), and $G-(A\cup(C-c'))$ contains a path from b to c' (and not using b' if $b\neq b'$).

Then we call $(G,(a,b,c),(a',b',c'),A,C)$ a *good* rung. Let $(L,(x_0,v_0,y_0),(x_m,v_m,y_m))$ be a ladder with good rungs $(R_i,(x_{i-1},v_{i-1},y_{i-1}),(x_i,v_i,y_i),X_i,Y_i)$, $i=1,\ldots,m$. Then we say that $(L,(x_0,v_0,y_0),(x_m,v_m,y_m),X,Y)$ is a *good* ladder along $v_0\ldots v_m$, where $X=\bigcup_{i=1}^m X_i$ and $Y=\bigcup_{i=1}^m Y_i$.

The following is the main result of [6].

(2.4) Lemma. *Let $(G,\{a,c\},\{a',c'\},(b,b'))$ be an obstruction satisfying the following conditions:*

(i) $G-\{b,b'\}$ contains disjoint paths A, C from a, c to a', c', respectively,

(ii) $A\cup C$ is an induced subgraph of G, and

(iii) $G-(A\cup C)$ is connected.

Then one of the following statements holds:

(1) There is a separation (G_1,G_2) in G of order at most 2 such that $\{a,b,c\}\subseteq V(G_1)$ and $\{a',b',c'\}\subseteq V(G_2)$.

(2) There is a subset $T\subseteq V(G)$ such that $|T|\leq 3$ and some component of $G-T$ contains no element of $\{a,b,c\}\cup\{a',b',c'\}$.

(3) $(G,(a,b,c),(a',b',c'),A,C)$ is a good ladder.

(4) G has a separation (J,L) such that $V(J\cap L)=\{w_0,\ldots,w_n\}$, $(J,w_0,\ldots,w_n)$ is 3-planar, $(L,(a,b,c),(a',b',c'),A,C)$ is a good ladder along a sequence $v_0\ldots v_m$, where $v_0=b$, $v_m=b'$, and $w_0\ldots w_n$ is the reduced sequence of $v_0\ldots v_m$.

We can now state and prove the technical lemma mentioned earlier. The conditions of the lemma arise in the proof of (1.3).

(2.5) Lemma. *Let $(L, (a, b, c), (a', b', c'), A, C)$ be a good ladder with good rungs $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$, where $A_i = A \cap R_i$, $C_i = C \cap C_i$, $a = x_0$, $a' = x_m$, $b = v_0$, $b' = v_m$, $c = y_0$, and $c' = y_m$. Let L' be a graph and let L^* be obtained from the disjoint union of L and L' by adding edges between $V(A \cup C)$ and $V(L')$ and between $V(A)$ and $V(C)$. Assume the following conditions hold. (We use N instead of N_{L^*} to denote neighborhood in L^*.)*

(a) *For any $i, j \in \{1, \ldots, m\}$ with $i \le j$, let $L_{i,j}$ denote the graph obtained from L^* by deleting all $R_k - (A_k \cup C_k \cup \{v_{i-1}, v_j\})$ with $k < i$ or $k > j$ and by deleting $v_0, \ldots, v_m$ except v_{i-1} and v_j. Then $(L_{i,j}, \{a, c\}, \{a', c'\}, (v_{i-1}, v_j))$ is an obstruction.*

(b) *If $q \in V(A)$ (respectively, $q \in V(C)$) such that L contains a path from q to $\{v_0, \ldots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \ldots, v_m\}$, then, for any component D of L', $N(D) \cap V(A) \subseteq aAq$ or $N(D) \cap V(A) \subseteq qAa'$ (respectively, $N(D) \cap V(C) \subseteq cCq$ or $N(D) \cap V(C) \subseteq qCc'$).*

(c) *If L^* contains paths from distinct $s, s' \in V(A)$ (in that order from a to a') to distinct $t, t' \in V(C)$ (in that order from c' to c), respectively, which are internally disjoint from L, then L contains no path from $V(sAs' - \{s, s'\}) \cup V(tCt' - \{t, t'\})$ to $\{v_0, \ldots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \ldots, v_m\}$.*

(d) *Suppose that L^* contains paths from $s, s' \in V(A)$ (in that order from a to a') to $t, t' \in V(C)$ (in that order from c to c'), respectively, which are internally disjoint from L. If L contains a path from $V(tCt' - \{t, t'\})$ (respectively, $V(sAs' - \{s, s'\})$) to $\{v_0, \ldots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \ldots, v_m\}$, then there is a vertex $q \in sAs'$ (respectively, $q \in tCt'$) such that, for every component D of L', $N(D) \cap V(A) \subseteq aAq$ or $N(D) \cap V(A) \subseteq qAa'$ (respectively, $N(D) \cap V(C) \subseteq cCq$ or $N(D) \cap V(C) \subseteq qCc'$).*

(e) *If R_i is not connected, but $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ (respectively, $\{y_{i-1}, v_{i-1}\} \cup \{y_i, v_i\}$) is contained in a component of R_i, then $v_{i-1} = v_i$.*

(f) *For any $T \subseteq V(L^*)$ with $|T| \le 3$, every component of $L^* - T$ contains a vertex in $V(A \cup C) \cup \{v_0, \ldots, v_m\}$.*

Then $(L^*, (a, b, c), (a', b', c'))$ is a ladder along a sequence $z_0 \ldots z_p$, and the reduced sequence of $z_0 \ldots z_p$ is the reduced sequence of $v_0 \ldots v_m$.

Proof. Suppose (2.5) fails. Choose L^*, L, A, C so that $|V(L^*)| + |E(L^*)|$ is minimum. Then it is easy to see that $\{a, b, c\}$ and $\{a', b', c'\}$ are independent sets in L^*. We proceed by proving Claims 1–6.

Claim 1. L^* does not contain any separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a'', v_k, c''\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, where $k \in \{0, \ldots, m\}$, $a'' \in V(A)$, and $c'' \in V(C)$.

Suppose that such a separation (G_1, G_2) does exist. Note that $\{a'', v_k, c''\} \neq \{a, b, c\}$ and $\{a'', v_k, c''\} \neq \{a', b', c'\}$; for otherwise, since $\{a, b, c\}$ and $\{a', b', c'\}$ are independent in L^*, $L^* - \{a'', v_k, c''\}$ has a component not containing any element of $V(A) \cup V(C) \cup \{v_0, \ldots, v_m\}$, contradicting (f).

For $i \in \{1, 2\}$, let $L_i = L \cap G_i$, $L'_i = L' \cap G_i$, $X_i = A \cap G_i$ and $Y_i = C \cap G_i$.

First, we claim that $(L_1, (a, b, c), (a'', v_k, c''), X_1, Y_1)$ is a good ladder along a sequence whose reduced sequence is the reduced sequence of $v_0 \ldots v_k$. This is clear if, for every $i \in \{1, \ldots, m\}$, either $R_i \subseteq L_1$ or $R_i \subseteq L_2$. So assume that p is minimum such that $R_p \not\subseteq L_1$ and $R_p \not\subseteq L_2$, and q is maximum such that $R_q \not\subseteq L_1$ and $R_q \not\subseteq L_2$. Then, for each $p \leq i \leq q$, $\{a_{i-1}, v_{i-1}, c_{i-1}\} \not\subseteq L_2$ and $\{a_i, v_i, c_i\} \not\subseteq L_1$, and hence, $R_i \not\subseteq L_1$ and $R_i \not\subseteq L_2$. Let $R_i^1 = R_i$ for $1 \leq i < p$; then $(R_i^1, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$ is a good rung. Now assume $i \in \{p, \ldots, q\}$. Because R_i has no 3-separation (R'_i, R''_i) such that $\{x_{i-1}, v_{i-1}, y_{i-1}\} \subseteq R'_i$ and $\{x_i, v_i, y_i\} \subseteq R''_i$, either $x_{i-1}Ax_i \subseteq aAa''$ and $y_{i-1}Cy_i \subseteq c''Cc'$ or $x_{i-1}Ax_i \subseteq a''Aa'$ and $y_{i-1}Cy_i \subseteq cCc''$. This implies that $a'' \in \{x_{p-1}, \ldots, x_q\}$ and $c'' \in \{y_{p-1}, \ldots, y_q\}$. Let $R_i^1 = (R_i \cap L_1) \cup \{c''\}$ if $x_{i-1}Ax_i \subseteq aAa''$, and let $R_i^1 = (R_i \cap L_1) \cup \{a''\}$ if $y_{i-1}Cy_i \subseteq cCc''$. Note that R_i^1 is not connected. Since $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$ is a good rung, it is easy to see that (by using (2.3)) if $x_{i-1}Ax_i \subseteq aAa''$ then $(R_i^1, (x_{i-1}, v_{i-1}, c''), (x_i, v_k, c''), A_i, \{c''\})$ is a good rung, and if $y_{i-1}Cy_i \subseteq cCc''$ then $(R_i^1, (a'', v_{i-1}, y_{i-1}), (a'', v_k, y_i), \{a''\}, C_i)$ is a good rung. Hence, $(L_1, (a, b, c), (a'', v_k, c''), X_1, Y_1)$ is a good ladder along a sequence whose reduced sequence is the reduced sequence of $v_0 \ldots v_k$.

Similarly, we can show that $(L_2, (a'', v_k, c''), (a', b', c'), X_2, Y_2)$ is a good ladder along a sequence whose reduced sequence is the reduced sequence of $v_k \ldots v_m$.

It is easy to see that, for $i \in \{1, 2\}$, G_i, L_i, L'_i, X_i, Y_i (as L^*, L, L', A, C, respectively) also satisfy (a)–(f) of (2.5) (with a'', v_k, c'' as a', b', c', respectively, when $i = 1$, and with a'', b'', c'' as a, b, c, respectively, when $i = 2$).

Since $|V(G_1)| + |E(G_1)| < |V(L^*)| + |E(G^*)|$, $(G_1, (a, b, c), (a'', v_k, c''))$ is a ladder along $z_0 \ldots z_q$, where $z_0 = v_0$ and $z_q = v_k$, and the reduced sequence of $z_0 \ldots z_q$ is the reduced sequence of $v_0 \ldots v_k$. Similarly, $(G_2, (a'', v_k, c''), (a', b', c'))$ is a ladder along $z_q \ldots z_p$, where $z_q = v_k$ and $z_p = v_m$, and the reduced sequence of $z_q \ldots z_p$ is the reduced sequence of $v_k \ldots v_m$. Hence, $(L^*, (a, b, c), (a', b', c'))$ is a ladder along $z_0 \ldots z_p$, where $z_0 = v_0 = b$ and $z_p = v_m = b'$, and the reduced sequence of $z_0 \ldots z_p$ is the reduced sequence of $v_0 \ldots v_m$. This is a contradiction.

Claim 2. $b \neq b'$ and $\{a,c\} \neq \{a',c'\}$.

Suppose $b = b'$ or $\{a,c\} = \{a',c'\}$. Then by Claim 1, $(L^*, (a,b,c), (a',b',c'))$ is a rung (as in (1) of (1.1)), and so, a ladder, contradicting the choice of L^*.

Claim 3. (1) If $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ is contained in a component of R_i, then for each $x \in \{x_{i-1}, x_i\}$ and for every component D of L', $N(D) \cap V(A) \subseteq aAx$ or $N(D) \cap V(A) \subseteq xAa'$. (2) If $\{y_{i-1}, v_{i-1}\} \cup \{y_i, v_i\}$ is contained in a component of R_i, then for each $y \in \{y_{i-1}, y_i\}$ and for every component D of L', $N(D) \cap V(C) \subseteq cCy$ or $N(D) \cap V(C) \subseteq yCc'$.

Assume that $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ is contained in a component of R_i. Because $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$ is a good rung, it follows from (1) or (2) of (2.3) that, for each $x \in \{x_{i-1}, x_i\}$, $R_i - ((A - x) \cup C)$ contains a path from x to $\{v_{i-1}, v_i\}$. Hence, (1) follows from (b) (with x as $q \in V(A)$). Similarly, we can prove (2).

Claim 4. If R_i is connected then $v_{i-1} = v_i$.

Suppose R_i is connected and $v_{i-1} \neq v_i$. Since $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$ is a good rung, it follows from (1) of (2.3) that $R_i - ((A_i - x_i) \cup (C_i - y_{i-1}))$ contains a $(\{v_{i-1}, x_i\}, \{y_{i-1}, v_i\})$-linkage X and $R_i - ((A_i - x_{i-1}) \cup (C_i - y_i))$ contains a $(\{v_{i-1}, y_i\}, \{x_{i-1}, v_i\})$-linkage Y.

Then L^* has no path from $aAx_i - x_i$ to $y_{i-1}Cc' - y_{i-1}$ internally disjoint from L. Otherwise, let P be a path from $p \in V(aAx_i - x_i)$ to $p' \in V(y_{i-1}Cc' - y_{i-1})$ internally disjoint from L. Let $K = aAp \cup P \cup p'Cc' \cup cCy_{i-1} \cup x_iAa' \cup X$. Then, $K \subseteq L_{i,i}$, and K contains disjoint paths from a, v_{i-1}, c to c', a', v_i, respectively, contradicting (a).

Similarly, we can use Y (instead of X) to show that L^* has no path from $x_{i-1}Aa' - x_{i-1}$ to $cCy_i - y_i$ internally disjoint from L.

Therefore, $\{x_{i-1}, v_{i-1}, y_{i-1}\} = \{a,b,c\}$; for otherwise, by Claim 3, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, y_{i-1}\}$, $\{a,b,c\} \subseteq G_1$, and $\{a',b',c'\} \subseteq G_2$, contradicting Claim 1. Similarly, we can prove $\{x_i, v_i, y_i\} = \{a',b',c'\}$. Hence by Claim 2, bb' is the reduced sequence of $v_0 \ldots v_m$.

If $L' = \emptyset$, then $L^* = R_i$, and so, $(L^*, (a,b,c), (a',b',c'))$ is a ladder along bb', contradicting the choice of L^*.

So assume that $L' \neq \emptyset$. Let D be a component of L'. By (f), $|N(D) \cap V(A)| \geq 2$ or $|N(D) \cap V(C)| \geq 2$. By symmetry, let $u, v \in V(A) \cap N(D)$ with $u \neq v$. By (b), L has no path from $V(uAv - \{u,v\})$ to $\{b,b'\}$ internally disjoint from $A \cup C$. Let $u', v' \in V(A)$ with $u'Av'$ maximal such that $uAv \subseteq u'Av'$ and L has no path from $V(u'Av' - \{u',v'\})$ to $\{b,b'\}$ internally disjoint from $A \cup C$. Then for any component D^* of L', $N(D^*) \subseteq u'Av'$ or $N(D^*) \cap V(u'Av' - \{u',v'\}) = \emptyset$. (For otherwise, let $u^*, v^* \in N(D^*) \cap V(A)$ such that a, u^*, u', v^*, v', a' occur on A in this order and $u^*, v^* \notin \{u',v'\}$. Then L has no path from $V(u^*Av' - \{u^*, v'\})$ to $\{b,b'\}$ internally disjoint from $A \cup C$, and so, $\{u^*, v'\}$ contradicts the choice of $\{u', v'\}$.) Also, L^* has no path from $V(u'Av' - \{u',v'\})$ to $V(C)$ internally disjoint from L; otherwise, by Claim 2, such a path would be from

$aAx_i - x_i$ to $y_{i-1}Cc' - y_{i-1}$ or from $cCy_i - y_i$ to $x_{i-1}Aa' - x_{i-1}$, and internally disjoint from L, a contradiction. Hence $G - \{u', v'\}$ has a component containing no vertex of $V(A \cup C) \cup \{v_0, \ldots, v_m\}$, contradicting (f).

Claim 5. No R_i is connected.

Suppose that R_i is connected. By Claim 4, $v_{i-1} = v_i$. By Claim 2, $v_{i-1} \neq b$ or $v_i \neq b'$. By symmetry, assume that $v_{i-1} \neq b$.

If L^* has no path from $aAx_{i-1} - x_{i-1}$ to $y_{i-1}Cc' - y_{i-1}$ internally disjoint from L, and L^* has no path from $cCy_{i-1} - y_{i-1}$ to $x_{i-1}Aa' - x_{i-1}$ internally disjoint from L, then by Claim 3, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, y_{i-1}\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

So assume (by the symmetry between x_{i-1} and y_{i-1}) that L^* has a path S from $s \in V(aAx_{i-1} - x_{i-1})$ to $s' \in V(y_{i-1}Cc' - y_{i-1})$ internally disjoint from L. Select S so that aAs is minimal. By $(1')$ of (2.3), R_i has a path from y_{i-1} to v_i internally disjoint from $A \cup C$. Hence for all $j < i$, $N(R_j - (A_j \cup C_j \cup \{v_{i-1}\})) \cap V(sAx_{i-1} - s) = \emptyset$; otherwise, $L_{j,i}$ has disjoint paths from a, v_{j-1}, c to c', a', v_i, respectively, contradicting (a). Also, by (c) and by Claim 3, L^* has no path from $cCy_{i-1} - y_{i-1}$ to $sAa' - s$ internally disjoint from L.

If $N(L - (A \cup C)) \cap aAs = \emptyset$ and L^* has no path from aAs to $cCy_{i-1} - y_{i-1}$ internally disjoint from L, then by Claim 3, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, v_{i-1}, y_{i-1}\}$, $\{a, b, c\} \subseteq G_1$ and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

So let $r \in V(aAs)$ with rAs minimal such that $r \in N(L - (A \cup C))$ or L^* has a path from r to $cCy_{i-1} - y_{i-1}$ internally disjoint from L.

If $r \in N(L - (A \cup C))$, then let $x'_{i-1} = r$. By (b) (with r as $q \in V(A)$ there), for every component D of L', $N(D) \cap V(A) \subseteq aAx'_{i-1}$ or $N(D) \cap V(A) \subseteq x'_{i-1}Aa'$. If L^* has a path from r to $r' \in V(cCy_{i-1} - y_{i-1})$, then by (d) (with r, r', s, s' as s, s', t, t', respectively), there is a vertex $x'_{i-1} \in rAs$ such that, for every component D of L', $N(D) \cap V(A) \subseteq aAx'_{i-1}$ or $N(D) \cap V(A) \subseteq x'_{i-1}Aa'$.

Hence by the choices of r and s and by Claim 3, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x'_{i-1}, v_{i-1}, y_{i-1}\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

Claim 6. A_i, C_i and $\{v_{i-1}, v_i\}$ are contained in three different components of R_i.

Suppose on the contrary that Claim 6 fails. By Claim 4 and by symmetry, assume that R_i is not connected, but $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ is contained in a component of R_i. Then $y_{i-1} = y_i$ (since $(R_i, \{x_{i-1}, y_{i-1}\}, \{x_i, y_i\}, (v_{i-1}, v_i))$ is a rung). By (e), $v_{i-1} = v_i$. Hence, $x_{i-1} \neq x_i$, and so, we have $x_{i-1} \neq a'$ and $x_i \neq a$. Moreover, either $\{x_{i-1}, v_{i-1}\} \neq \{a, b\}$ or $\{x_i, v_i\} \neq \{a', b'\}$; otherwise, $b = v_{i-1} = v_i = b'$, contradicting Claim 2. By symmetry, assume that $\{a, b\} \neq \{x_{i-1}, v_{i-1}\} \neq \{a', b'\}$.

If $i = 1$ then define $B = \{v_0\}$, and if $i \geq 2$ then define $B = \bigcup_{1 \leq k \leq i-1}(R_k - (A_k \cup C_k))$. Let $B' = \bigcup_{k=i}^{m}(R_k - (A_k \cup C_k))$. Then $B \cap B' = \{v_{i-1}\}$, $N(B) \subseteq aAx_{i-1} \cup cCy_{i-1}$, and

$N(B') \subseteq x_{i-1}Aa' \cup y_{i-1}Cc'$.

Define $s \in V(C)$ with sCc' minimal such that $s = c$, or $s \in N(B - v_{i-1})$, or L^* has a path from $aAx_{i-1} - x_{i-1}$ to s internally disjoint from L. Define $t \in V(C)$ with cCt minimal such that $t = c'$, or $t \in N(B' - v_{i-1})$, or L^* contains a path from $x_{i-1}Aa' - x_{i-1}$ to t internally disjoint from L. We distinguish three cases. Note that by (2) of (2.3), $R_i - ((A_i - x_{i-1}) \cup C_i)$ has a path from x_{i-1} to v_i.

Case 1. $s = c$ or $t = c'$.

Suppose $s = c$ (respectively, $t = c'$). Then by Claim 3 and by the choice of s (respectively, t), L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, c\}$ (respectively, $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, c'\}$), $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

Case 2. $t \neq c'$ and $s \in N(B - v_{i-1})$.

Then $N(B') \cap V(cCs - s) = \emptyset$ because $N(B') \cap V(C) \subseteq y_{i-1}Cc'$ and $N(B) \cap V(C) \subseteq cCy_{i-1}$. By (b) (with s as $q \in V(C)$), for every component D of L', $N(D) \cap V(C) \subseteq cCs$ or $N(D) \cap V(C) \subseteq sCc'$. Let $s \in N(R_l - (A_l \cup C_l \cup \{v_l\}))$ for some $l \leq i-1$.

Hence, L^* does not contain any path from $x_{i-1}Aa' - x_{i-1}$ to $cCs - s$; otherwise, $L_{l,i}$ contains three disjoint paths from a, v_{l-1}, c to $v_{i-1} = v_i, c', a'$, respectively, contradicting (a).

By the choice of s and by Claim 3, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, s\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

Case 3. $t \neq c'$ and L^* has a path S from $aAx_{i-1} - x_{i-1}$ to s internally disjoint from L.

If L^* has a path from $x_{i-1}Aa' - x_{i-1}$ to t internally disjoint from L, then $t \in sCc'$ by (c) (with s, t as t, t' in (c), respectively). By (d) (with s, t as t, t' in (d), respectively), there is some vertex $q \in sCt$ such that, for every component D of L', $N(D) \cap V(C) \subseteq cCq$ or $N(D) \cap V(C) \subseteq qCc'$. So by Claim 3 and by the choices of s and t, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, q\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1.

Now assume that $t \in N(B' - v_{i-1})$. Then by (b) (with t as $q \in V(C)$), for every component D of L', $N(D) \cap V(C) \subseteq cCt$ or $N(D) \cap V(C) \subseteq tCc'$. Let $t \in N(R_l - (A_l \cup C_l \cup \{v_{l-1}\}))$ for some l with $i \leq l \leq m$. Hence, $t \in sCc'$; otherwise, $L_{i,l}$ contains three disjoint paths from a, v_{i-1}, c to c', a', v_l, respectively, contradicting (a). Thus, by Claim 3 and by the choices of s and t, L^* has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_{i-1}, v_{i-1}, t\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1. This proves Claim 6.

By Claim 6, A, C and $\{v_0, \ldots, v_m\}$ are in different components of L. Hence, $A \cup C$ and $\{b, b'\}$ are contained in different components of L^*. Since $b \neq b'$ (by Claim 2), G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, b', c\}$, $\{a, b, c\} \subseteq G_1$, and $\{a', b', c'\} \subseteq G_2$, contradicting Claim 1. □

3 The proof of main result

First, we prove the following lemma, which proves the necessary part of (1.3).

(3.1) Lemma. *Let $(G, \{a, c\}, \{a', c'\}, (b, b'))$ be an obstruction. Then one of the following statements holds.*

(1) G has a separation (G_1, G_2) of order at most 2 such that $\{a, b, c\} \subseteq V(G_1)$ and $\{a', b', c'\} \subseteq V(G_2)$.

(2) There is a subset $T \subseteq V(G)$ such that $|T| \leq 3$ and some component of $G - T$ contains no element of $\{a, b, c\} \cup \{a', b', c'\}$.

(3) $(G, (a, b, c), (a', b', c'))$ is a ladder.

(4) G has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$.

Proof. Assume that (1) and (2) do not hold. Let A, B, C be disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$ in G such that $a \in A$, $c \in C$, and B is from b to b'. We choose A, B and C to be induced paths. Let H' denote the component of $G - (A \cup C)$ containing B. Let K be the subgraph of G obtained from $A \cup C \cup H'$ by adding those edges in G between $V(A) \cup V(C)$ and $V(H')$. It is straightforward to verify that A, B, C, H' and K satisfy the conditions (i)–(iii) of (2.4) (with K as G there). By applying (2.4), one of (1)–(4) of (2.4) holds.

Because of A, B, C, (1) of (2.4) does not hold. Since we assume that (2) of (3.1) does not hold, if there is some $T \subseteq V(K)$ with $|T| \leq 3$ such that some component U of $G - T$ contains no element of $\{a, b, c\} \cup \{a', b', c'\}$, then T would be contained in the "ladder part" of K when (3) or (4) of (2.4) holds (see the comment after (1.3)). So we may assume that either (1a) $(K, (a, b, c), (a', b', c'), A, C)$ (when $a' \in A$ and $c' \in C$) or $(K, (a, b, c), (c', b', a'), A, C)$ (when $a' \in C$ and $c' \in A$) is a good ladder along a sequence $v_0 \ldots v_m$, where $v_0 = b$ and $v_m = b'$ (in this case, let $J = \{w_0, \ldots, w_n\}$ and $L = K$, where $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$), or (1b) K has a separation (J, L) such that $V(J \cap L) = \{w_0, \ldots, w_n\}$, $(J, w_0, \ldots, w_n)$ is 3-planar, $(L, (a, b, c), (a', b', c'), A, C)$ (when $a' \in A$ and $c' \in C$) or $(L, (a, b, c), (c', b', a'), A, C)$ (when $a' \in C$ and $c' \in A$) is a good ladder along $v_0 \ldots v_m$, where $v_0 = b$, $v_m = b'$, and $w_0 \ldots w_n$ is the reduced sequence of $v_0 \ldots v_m$.

Let $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i), A_i, C_i)$, $i = 1, \ldots, m$, be the good rungs of L, where $A_i = A \cap R_i$, $C_i = C \cap R_i$, $a = x_0$, $c = y_0$, $a' = x_m$ (when $a' \in A$) or $c' = x_m$ (when $c' \in A$), and $c' = y_m$ (when $c' \in C$) or $a' = y_m$ (when $a' \in C$). Let $v(L) = \{x \in V(R_k - v_k) : v_{k-1} = v_k, 1 \leq k \leq m\}$. We select A, B, C, H', J and L such that

(I) H' is maximal,

(II) subject to (I), $v(L)$ is maximal, and

(III) subject to (II), J is maximal.

Without loss of generality, we may choose the notation so that A, B, C are from a, b, c to a', b', c', respectively. Let $L' = G - K$ and let $L^* = G[V(L \cup L')]$. Then L^* is obtained from the disjoint union of L and L' by adding edges between $V(A) \cup V(C)$ and $V(L')$ or between $V(A)$ and $V(C)$. Next we will show that L^*, L, L', A, C satisfy the remaining conditions (a)–(f) of (2.5).

Note that since (2) does not hold for G, J can be drawn in a closed disc in the plane with no edge crossings such that $w_0, w_1, \ldots, w_n$ occur on the boundary of the disc in that cyclic order.

(a) For any $i, j \in \{1, \ldots, m\}$ with $i \leq j$, let $L_{i,j}$ denote the graph obtained from L^* by deleting all $R_k - (A_k \cup C_k \cup \{v_{i-1}, v_j\})$ with $k < i$ or $k > j$ and by deleting $v_0, \ldots, v_m$ except v_{i-1} and v_j. Then $(L_{i,j}, \{a, c\}, \{a', c'\}, (v_{i-1}, v_j))$ is an obstruction.

Clearly (a) holds when $v_{i-1} = v_j$. So assume that $v_{i-1} \neq v_j$. For convenience, let G^* denote the union of J and all $R_k - (A_k \cup C_k)$ with $k < i$ or $k > j$. Then $V(G^* \cap L_{i,j}) = \{v_{i-1}, v_j\}$.

If we let $\mathcal{J} = \{V(R_k) - (V(A_k \cup C_k) \cup \{v_{k-1}, v_k\}) : k < i \text{ or } k > j\}$, then $(G^*, \mathcal{J}, v_0, \ldots, v_{i-1}, v_j, \ldots, v_m)$ is 3-planar. Hence, let $\mathcal{J}^*$ be a collection of pairwise disjoint subsets of $V(J) - \{v_0, \ldots, v_{i-1}, v_j, \ldots, v_m\}$ such that $(G^*, \mathcal{J}^*, v_0, \ldots, v_{i-1}, v_j, \ldots, v_m)$ is 3-planar and $\mathcal{J}^*$ is minimal.

If G^* contains a $(\{b, v_{i-1}\}, \{b', v_j\})$-linkage, then $(L_{i,j}, \{a, c\}, \{a', c'\}, (v_{i-1}, v_j))$ is an obstruction. For otherwise, $L_{i,j}$ has three disjoint paths form $\{a, v_{i-1}, c\}$ to $\{a', v_j, c'\}$, none from v_{i-1} to v_j. These three paths in $L_{i,j}$ combined with a $(\{b, v_{i-1}\}, \{b', v_j\})$-linkage in G^* give three disjoint paths in G from $\{a, b, c\}$ to $\{a', b', c'\}$, none from b to b'. This contradicts the assumption that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction.

So assume that G^* contains no $(\{b, v_{i-1}\}, \{b', v_j\})$-linkage. Then $p(G^*, \mathcal{J}^*)$ has no $(\{b, v_{i-1}\}, \{b', v_j\})$-linkage; otherwise, by (2.2) (with $G^*, \mathcal{J}^*, v_{i-1}, v_j$ as $G, \mathcal{A}, u, v$, respectively), G^* would contain a $(\{b, v_{i-1}\}, \{b', v_j\})$-linkage. By planarity, $p(G^*, \mathcal{J}^*)$ has a separation (G_1', G_2') such that $|V(G_1' \cap G_2')| \leq 1$, $\{v_{i-1}, v_j\} \subseteq G_1'$, and $\{b, b'\} \subseteq G_2'$. Again by planarity of $p(G^*, \mathcal{J}^*)$, there are $r \leq i-1$ and $s \geq j$ such that $\{v_r, \ldots, v_s\} \subseteq G_1'$ and $\{v_0, \ldots, v_{r-1}, v_{s+1}, \ldots, v_m\} \subseteq G_2'$. Let G_i denote the subgraph of G^* induced by the vertices contained in $V(G_i')$ or contained in some $U \in \mathcal{J}^*$ with $N_{G^*}(U) \subseteq V(G_i')$. Then (G_1, G_2) is a separation of G^* such that $|V(G_1 \cap G_2)| \leq 1$, $\{v_r, \ldots, v_s\} \subseteq G_1$, and $\{v_0, \ldots, v_{r-1}, v_{s+1}, \ldots, v_m\} \subseteq G_2$.

Because $K - (A \cup C)$ is connected, $|V(G_1 \cap G_2)| = 1$. Let w be the vertex in $V(G_1 \cap G_2)$. Then K has a separation (M_1, M_2) such that $V(M_1 \cap M_2) = \{x_{r-1}, y_{r-1}, w, x_s, y_s\}$, $M_1 = G_1 \cup (\bigcup_{r \leq k \leq s} R_k)$, and $G_2 \subseteq M_2$.

Let $a_0, \dots, a_l$ be vertices in that order on A and $c_0, \dots, c_l$ be vertices in that order on C such that (a1) $a_0 = x_{r-1}$, $a_l = x_s$, $c_0 = y_{r-1}$, and $c_l = y_s$, (a2) for $i = 1, \dots, l-1$, $\{a_i, c_i\}$ is a 2-cut of $M_1 - w$, and (a3) for any $x \in V(a_{i-1}Aa_i)$ and $y \in V(c_{i-1}Cc_i)$, $\{x, y\}$ is not a 2-cut of $M_1 - w$ unless $\{x, y\} = \{a_{i-1}, c_{i-1}\}$ or $\{x, y\} = \{a_i, c_i\}$. Let S denote the set of edges of M_1 with both ends contained in $\{a_k, c_k, w\}$, $k = 0, \dots, l$.

Let M_1^k denote the subgraph of $M_1 - S$ induced by those vertices x such that every path in $M_1 - w$ from x to $\{x_{r-1}, y_{r-1}, w, x_s, y_s\}$ intersect $\{a_{k-1}, a_k, w, c_{k-1}, c_k\}$. Because (2) does not hold for G and by (a3), M_1^k has no 3-separation (M', M'') such that $\{a_{k-1}, w, c_{k-1}\} \subseteq M'$ and $\{a_k, w, c_k\} \subseteq M''$. By (1) of (1.1), $(M_1^k, (a_{k-1}, w, c_{k-1}), (a_k, w, c_k))$ is a rung. In fact, by (a3), it is easy to show that $(M_1^k, (a_{k-1}, w, c_{k-1}), (a_k, w, c_k), a_{k-1}Aa_k, c_{k-1}Cc_k)$ is a good rung.

Again because (2) does not hold for G and by (a3), $M_1 = (\bigcup_{k=1}^{l} M_1^k) + S$. Therefore, we see that $(M_1, (x_{r-1}, v_r, y_{r-1}), (x_s, v_s, y_s), A \cap M_1, C \cap M_1)$ is a good ladder along w.

Let $L_1 = M_1 \cup L$, and let $J_1 = J - (G_1 - w)$. Then (J_1, L_1) is a separation of K such that $V(J_1 \cap L_1) = \{w_0, \dots, w_{t-1}, w, w_t, \dots, w_n\}$ for some appropriate t, $(J_1, w_0, \dots, w_{t-1}, w, w_t, \dots, w_n)$ is 3-planar, $(L_1, (a, b, c), (a', b', c'), A, C)$ is a good ladder along a sequence whose reduced sequence is $w_0 \dots w_{t-1} w w_t \dots w_n$. Note that H' remains unchanged, but $v(L) \cup (\{v_{i-1}, v_j\} - \{w\}) \subseteq v(L_1)$, and $v_{i-1}, v_j \notin v(L)$, contradicting (II).

(b) If $q \in V(A)$ (respectively, $q \in V(C)$) such that L contains a path from q to $\{v_0, \dots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \dots, v_m\}$, then for every component D of L', $N(D) \cap V(A) \subseteq aAq$ or $N(D) \cap V(A) \subseteq qAa'$ (respectively, $N(D) \cap V(C) \subseteq cCq$ or $N(D) \cap V(C) \subseteq qCc'$).

Suppose (b) is false. By symmetry, assume that L contains a path from $q \in V(A)$ to $\{v_0, \dots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \dots, v_m\}$, and assume that there is a component D of L' such that $N(D) \cap V(aAq - q) \neq \emptyset$ and $N(D) \cap V(qAa' - q) \neq \emptyset$. Let $u \in N(D) \cap V(aAq - q)$ and $v \in N(D) \cap V(qAa' - q)$.

Let A' be an induced path from a to a' in L^* such that $V(A') \subseteq aAu \cup D \cup vAa'$. Then the component of $G - (A' \cup C)$ containing B is larger than H', contradicting (I).

(c) If L^* contains paths from distinct $s, s' \in V(A)$ (in that order from a to a') to distinct $t, t' \in V(C)$ (in that order from c' to c), respectively, which are internally disjoint from L, then L contains no path from $V(sAs' - \{s, s'\}) \cup V(tCt' - \{t, t'\})$ to $\{v_0, \dots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \dots, v_m\}$.

Let P, P' be the paths in (c) from s, s' to t, t', respectively, which are internally disjoint from L. If $P \cap P' \neq \emptyset$, then there is a component D of L' such that $\{s, s', t, t'\} \subseteq N(D)$, and (c) follows from (b). So assume that $P \cap P' = \emptyset$.

Suppose that (c) fails, and assume by symmetry that L^* has a path from $V(sAs' - \{s, s'\})$ to $\{v_0, \dots, v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0, \dots, v_m\}$. Let A' be an induced path in L^* from a to c' such that $V(A') \subseteq aAs \cup P \cup tCc'$, and let C' be an induced path in L^* from c to a' such that $V(C') \subseteq cCt' \cup P' \cup s'Aa'$. Then the component of $G-(A' \cup C')$ containing B is larger than H', contradicting (I).

(d) Suppose that L^* contains paths from $s, s' \in V(A)$ (in that order from a to a') to $t, t' \in V(C)$ (in that order from c to c') which are internally disjoint from L. If L contains a path from $V(tCt' - \{t,t'\})$ (respectively, $V(sAs' - \{s,s'\})$) to $\{v_0,\dots,v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0,\dots,v_m\}$, then there is a vertex $q \in sAs'$ (respectively, $q \in tCt'$) such that, for every component D of L', $N(D) \cap V(A) \subseteq aAq$ or $N(D) \cap V(A) \subseteq qAa'$ (respectively, $N(D) \cap V(C) \subseteq cCq$ or $N(D) \cap V(C) \subseteq qCc'$).

By symmetry, assume L contains a path from $V(tCt' - \{t,t'\})$ to $\{v_0,\dots,v_m\}$ internally disjoint from $V(A \cup C) \cup \{v_0,\dots,v_m\}$. Let $r, r' \in V(A)$ with $rAr' \subseteq sAs'$ and rAr' minimal such that L^* has paths R, R' from $r, r' \in V(A)$ to t, t', respectively, which are internally disjoint from L. By (c), a, s, r, r', s', a' occur on A in that order. We will show that there is some $q \in V(rAr')$ such that, for every component D of L', $N(D) \cap V(A) \subseteq aAq$ or $N(D) \cap V(A) \subseteq qAa'$.

We may assume that L' has a component D_1 such that $N(D_1) \cap V(aAr - r) \neq \emptyset$ and $N(D_1) \cap V(rAa' - r) \neq \emptyset$; otherwise, $q := r$ would be the desired vertex. Let $p_1 \in V(aAr - r) \cap N(D_1)$ and $q_1 \in V(rAa' - r) \cap N(D_1)$, and let P_1 be a path in L^* from p_1 to q_1 through D_1 and internally disjoint from L. Select D_1, P_1, p_1, q_1 so that q_1Aa' is minimal and subject to this, p_1Aq_1 is maximal. See Figure 1.

Then $q_1 \in rAr' - r$. For otherwise, $q_1 \in r'Ac' - r'$. By (c), $D_1 \cap (R \cup R') = \emptyset$. Let A' be an induced path in L^* from a to a' such that $V(A') \subseteq aAp_1 \cup P_1 \cup q_1Aa'$, and let C' be an induced path from c to c' in L^* such that $V(C') \subseteq cCt \cup R \cup rAr' \cup R' \cup t'Cc'$. Then the component of $G - (A' \cup C')$ containing B is larger than H', contradicting (I).

By (c), $R' \cap D_1 = \emptyset$. By the minimality of rAr', $R \cap D_1 = \emptyset$. We may assume that there are $p_2 \in V(aAq_1 - q_1)$ and $q_2 \in V(q_1Aa' - q_1)$ such that $\{p_2, q_2\} \subseteq N(D_2)$ for some component D_2 of L'; otherwise, $q := q_1$ would be the desired vertex. Let P_2 be a path in L^* from p_2 to q_2 through D_2 and internally disjoint from L. Select D_2, P_2, p_2, q_2 such that q_2Aa' is minimal, and subject to this, p_2Aq_2 is maximal. By the choice of D_1, P_1, p_1 and q_1, we conclude that $p_2 \in rAq_1 - q_1$. If $q_2 \in r'Aa' - r'$, then we stop this process; and if $q_2 \in q_1Ar' - q_1$, then we find D_3, P_3, p_3, q_3 in a similar way. Since G is a finite graph, we have a sequence $P_1, P_2, \dots, P_l$ of paths with P_i from p_i to q_i such that $q_l \in r'Aa' - r'$, and for $i \in \{1, \dots, l-1\}$, $p_{i+1} \in q_{i-1}Aq_i - q_i$ and $q_i \in p_{i+1}Ap_{i+2} - p_{i+1}$, where $q_0 = r$ and $p_{l+1} = r'$. Figure 1 illustrates the cases for $l = 3$ and $l = 4$.

By (c) and by the minimality of rAr', $(P_i - \{p_i, q_i\}) \cap (R \cup R') = \emptyset$ for $i \in \{1, \dots, l\}$. By the choice of P_i, all P_i's are internally disjoint, and $P_i \cap P_j = \emptyset$ if $|i - j| \neq 2$.

If l is odd, then let A' be an induced path in L^* from a to a' such that $V(A') \subseteq aAp_1 \cup P_1 \cup q_1Ap_3 \cup P_3 \cup q_3Ap_5 \cup \ldots \cup P_l \cup q_lAa'$, and let C' be an induced path in L^* from c

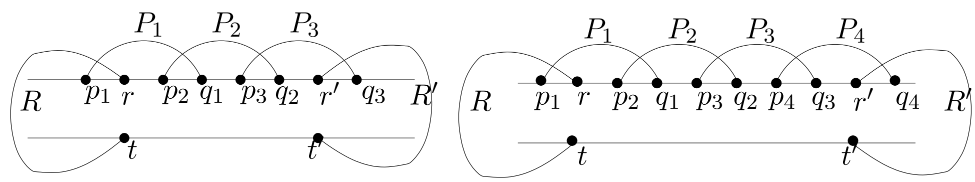

Figure 1: Paths $P_1, \dots, P_l$.

to c' such that $V(C') \subseteq cCt \cup R \cup rAp_2 \cup P_2 \cup q_2Ap_4 \cup P_4 \cup \ldots \cup P_{l-1} \cup q_{l-1}Ar' \cup R' \cup t'Cc'$. If l is even, then let A' be an induced path in L^* from a to c' such that $V(A') \subseteq aAp_1 \cup P_1 \cup q_1Ap_3 \cup P_3 \cup q_3Ap_5 \cup \ldots \cup P_{l-1} \cup q_{l-1}Ar' \cup R' \cup t'Cc'$, and let C' be an induced path in L^* from c to a' such that $V(C') \subseteq cCt \cup R \cup rAp_2 \cup P_2 \cup q_2Ap_4 \cup P_4 \cup \ldots \cup P_l \cup q_lAa'$. Clearly, the component of $G - (A' \cup C')$ containing B is larger than H', contradicting (I).

(e) If R_i is not connected and $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ (respectively, $\{y_{i-1}, v_{i-1}\} \cup \{y_i, v_i\}$) is contained in a component of R_i, then $v_{i-1} = v_i$.

Suppose $v_{i-1} \neq v_i$. Assume by symmetry that $\{x_{i-1}, v_{i-1}\} \cup \{x_i, v_i\}$ is contained in a component U of R_i. Then $y_{i-1} = y_i$ and $|V(U)| \geq 4$ because $E(U) \neq \emptyset$ and R_i has no 3-separation (R', R'') such that $\{x_{i-1}, v_{i-1}, y_{i-1}\} \subseteq R'$ and $\{x_i, v_i, y_i\} \subseteq R''$. By assumption that (2) does not hold for G, $x_{i-1} \neq x_i$;

Moreover, there is a collection $\mathcal{U}$ of pairwise disjoint subsets of $V(U) - \{x_{i-1}, v_{i-1}, x_i, v_i\}$ such that $(U, \mathcal{U}, x_{i-1}, x_i, v_i, v_{i-1})$ is 3-planar; otherwise, U contains an $(\{x_{i-1}, v_i\}, \{v_{i-1}, x_i\})$-linkage (by (2.1)), and so, $L_{i,i}$ has disjoint paths from a, v_{i-1}, c to v_i, a', c', respectively, contradicting (a).

Note that $U - A_i$ is connected. We claim that $\{v_{i-1}, v_i\}$ is contained in a block M of $U - A_i$. For otherwise, let (U_1, U_2) be a 1-separation in $U - A_i$ such that $v_{i-1} \in U_1$ and $v_i \in U_2$. Then there are $u_j \in V(A_i) \cap N(U_j)$, $j = 1, 2$, such that x_{i-1}, u_2, u_1, x_i occur on A in that order; as otherwise U has a 2-separation (U_1', U_2') such that $U_1 \cup \{x_{i-1}\} \subseteq U_1'$ and $U_2 \cup \{x_i\} \subseteq U_2'$, and $(U_1' \cup \{y_i\}, U_2' \cup \{y_i\})$ would be a 3-separation in R_i, a contradiction. Hence, U has an $(\{x_{i-1}, v_i\}, \{v_{i-1}, x_i\})$-linkage. This implies that $L_{i,i}$ has disjoint paths from a, v_{i-1}, c to v_i, a', c', respectively, contradicting (a).

We claim that $X \cap V(M) = \emptyset$ for all $X \in \mathcal{U}$. Otherwise, let $X \in \mathcal{U}$ such that $X \cap V(M) \neq \emptyset$. Then $|N_U(X) \cap V(M)| \geq 2$ because M is 2-connected. Hence, $|N_U(X) \cap V(A)| \leq 1$. Therefore $G - N_U(X)$ has a component containing no element of $\{a, b, c\} \cup \{a', b', c'\}$, contradicting the assumption that (2) does not hold for G.

Thus M is a plane subgraph of $p(U, \mathcal{U})$. Let $b_0, \dots, b_k$ be the vertices on the outer cycle of M (in that clockwise order from $b_0 = v_{i-1}$ to $b_k = v_i$) such that, for each $j \in \{1, \dots, k-1\}$, $b_j \in N(A)$ or b_j is a cut vertex of $U - A$. Let $R_i' = R_i - (M - \{b_0, \dots, b_k\})$.

By 3-planarity of U, it is easy to verify that $(R_i', (x_{i-1}, v_{i-1}, y_i), (x_i, v_i, y_i))$ is a ladder along a sequence whose reduced sequence is $b_0 \dots b_k$. In fact, it is easy to see that $(R_i', (x_{i-1}, v_{i-1}, y_i), (x_i, v_i, y_i), A_i, C_i)$ is a good ladder (its rungs are not connected).

Let $J_1 = J \cup M$ and let $L_1 = L - (M - \{b_0, \dots, b_k\})$. Then (J_1, L_1) is a separation in K such that $V(L_1 \cap J_1) = \{w_0, \dots, w_n\} \cup \{b_0, \dots, b_k\}$, $(L_1, w_0, \dots, w_{j-1}, b_0, \dots, b_k, w_j, \dots, w_n)$ is 3-planar (for some appropriate j such that $w_{j-1} = v_r$ and $w_j = v_s$ for some $r \leq i-1$ and $s \geq i$), $(L_1, (a,b,c), (a',b',c'), A, C)$ is a good ladder along a sequence S, where $w_0 \dots w_{j-1} b_0 \dots b_k w_j \dots w_n$ is the reduced sequence of S. Note that H' remains the same and $v(L) \subseteq v(L_1)$ (since $v_{i-1} \neq v_i$), but J_1 contains J as a proper subgraph, contradicting (III). This proves (e).

Because (2) does not hold for G, we have

(f) For any $T \subseteq V(L^*)$ with $|T| \leq 3$, every component of $L^* - T$ contains a vertex in $V(A \cup C) \cup \{v_0, \dots, v_m\}$.

By (a)–(f), L^*, L, L', A, C satisfy all conditions of (2.5). Hence, by (2.5), $(L^*, (a,b,c), (a',b',c'))$ is a ladder along a sequence $z_0 z_1 \dots z_p$ whose reduced sequence is the reduced sequence of $v_0 \dots v_m$. Hence $w_0 \dots w_n$ is the reduced sequence of $z_0 \dots z_p$. Therefore, either $G = L^*$ or G has a separation (J, L^*) such that $V(J \cap L^*) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, and $(L^*, (a,b,c), (a',b',c'))$ is a ladder along $z_0 \dots z_p$, where $z_0 = b$, $z_p = b'$, $w_0 \dots w_n$ is the reduced sequence of $z_0 \dots z_p$. □

Proof of (1.3). Suppose $(G, \{a,c\}, \{a',c'\}, (b,b'))$ is an obstruction, and assume that, for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G - T$ contains an element of $\{a,b,c\} \cup \{a',b',c'\}$. Then by (3.1), (1), (2) or (3) of (1.3) holds.

Now assume that (1), (2) or (3) of (1.3) holds. We wish to show that $(G, \{a,c\}, \{a',c'\}, (b,b'))$ is an obstruction.

If (1) of (1.3) holds, then $(G, \{a,c\}, \{a',c'\}, (b,b'))$ is an obstruction in the trivial sense. If (2) of (1.3) holds, then $(G, (a,b,c), (a',b',c'))$ is a ladder, and so, an obstruction. So we may assume that (1) and (2) of (1.3) do not hold, and (3) of (1.3) holds. Then G contains three disjoint paths from $\{a,b,c\}$ to $\{a',b',c'\}$. Let A, B, C be disjoint paths in G from $\{a,b,c\}$ to $\{a',b',c'\}$, with $b \in B$. We may assume that A, B, C are chosen so that $b' \notin B$; for otherwise, $(G, (a,b,c), (a',b',c'))$ is an obstruction.

Let (J, L) be a separation of G such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, and $(L, (a,b,c), (a',b',c'))$ is a ladder with rungs $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i = 1, \dots, m$, where $x_0 = a$, $v_0 = b$, $y_0 = c$, $x_m = a'$, $v_m = b'$, $y_m = c'$, and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$. We use induction on the number of rungs in L.

Since $b' \notin B$, $B \not\subseteq J$. Hence, let $w_k \in B$ with k maximum such that $bBw_k \subseteq J$. Then $k < n$; otherwise, $b' \in B$ (since $(J, w_0, \dots, w_n)$ is 3-planar). Let $v_s = w_k$ with s minimum and let $v_t = w_{k+1}$ with t maximum. Let L^* be the union of $\bigcup_{s<i\leq t} R_i$ and those edges of L with both ends in some $\{x_i, v_i, y_i\}$, $s < i \leq t$. Then $(L^*, (x_s, v_s, y_s), (x_t, v_t, y_t))$ is a ladder along $v_s \dots v_t$. Since every path in G from $\{a, b, c\}$ to $\{a', b', c'\}$ must intersect $V(bBw_k) \cup \{x_s, y_s\}$, $A \cap L^*$, $B \cap L^*$, and $C \cap L^*$ are disjoint paths from $\{x_s, w_k, y_s\}$ to $\{x_t, w_{k+1}, y_t\}$. Since $(L^*, (x_s, v_s, y_s), (x_t, v_t, y_t))$ is a ladder, $B \cap L^*$ is a path from w_k to w_{k+1}.

Let L' denote the union of $\bigcup_{j=t+1}^{m} R_j$ and edges in L with both ends in some $\{x_j, v_j, y_j\}$, $j = t+1, \dots, m$. Then $(L', (x_t, v_t, y_t), (a', b', c'))$ is a ladder along $v_t \dots v_m$. Let $J' = J + w_k w_{k+1}$, and let the edge $w_k w_{k+1}$ be added such that $(J', w_0, \dots, w_n)$ is 3-planar. It is easy to see that $A \cap (J' \cup L')$, $(B \cap (J' \cup L')) + w_k w_{k+1}$, $C \cap (J' \cup L')$ are disjoint paths in $J' \cup L'$ from $\{b, x_t, y_t\}$ to $\{a', b', c'\}$. Note that (J', L') is a separation in $J' \cup L'$ such that $V(J' \cap L') = \{b, w_{k+1}, \dots, w_n\}$, $(J', b, w_{k+1}, \dots, w_n)$ is 3-planar, and $(L', (x_t, b, y_t), (a', b', c'))$ is a ladder along $bv_t \dots v_m$, where $v_m = w_n = b'$, and $bw_{k+1} \dots w_n$ is the reduced sequence of $bv_t \dots v_m$. Since L' has fewer rungs than L, it follows from induction that $(B \cap (J' \cup L')) + w_k w_{k+1}$ is from b to b'. Hence, B is from b to b'. So $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction. □

4 Connectivity

In this section, we prove (1.4), and give an example to show that (1.4) is best possible.

Proof of (1.4). Suppose that G is 8-connected and $\{a, b, c, a', b', c'\} \subseteq V(G)$, and assume that $(G, \{a, c\}, \{a', c'\}, (b, b'))$ is an obstruction. By (1.3), either (i) $(G, (a, b, c), (a', b', c'))$ is a ladder along a sequence $v_0 \dots v_m$ (in this case, let $J = \{w_0, \dots, w_n\}$ and $L = G$, where $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$), or (ii) G has a separation (J, L) such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, and $(L, (a, b, c), (a', b', c'))$ is a ladder with rungs along $v_0 \dots v_m$, where $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.

Then $L - \{w_0, \dots, w_n\} \neq \emptyset$; otherwise, $G = J$ is 3-planar, and so, is at most 5-connected, a contradiction. Since $(J, w_0, \dots, w_n)$ is 3-planar and since G is 8-connected, J is a plane graph. Let $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$, $i = 1, \dots, m$, be the rungs of L, where $x_0 = a$, $v_0 = b$, $y_0 = c$, $x_m = a'$, $v_m = b'$, and $y_m = c'$.

(a) We claim that L is at most 6-connected.

This is obvious when $m \geq 2$. So assume that $m = 1$. Then $(L, (a, b, c), (a', b', c'))$ is a rung. Since $\{a, c\} \neq \{a', c'\}$ and $b \neq b'$, L is at most 6-connected.

By (a) and since G is 8-connected, $J - \{w_0, \dots, w_n\} \neq \emptyset$, $n \geq 7$, and $V(R_i) = \{x_{i-1}, y_{i-1}, v_{i-1}\} \cup \{x_i, y_i, v_i\}$ for $i \in \{1, \dots, m\}$.

(b) We claim that $v_0 \neq v_1$ and $\{x_1, y_1\} = \{x_0, y_0\}$, and $v_{m-1} \neq v_m$ and $\{x_m, y_m\} = \{x_{m-1}, y_{m-1}\}$.

If $\{x_1, y_1\} \neq \{x_0, y_0\}$, then $\{v_0, x_1, v_1, y_1\}$ is a cut set of G, a contradiction. So $\{x_1, y_1\} = \{x_0, y_0\}$, and hence, $v_1 \neq v_0$.

Similarly, $v_{m-1} \neq v_m$ and $\{x_m, y_m\} = \{x_{m-1}, y_{m-1}\}$.

(c) We claim that, for every $x \in V(J) - \{w_1, \ldots, w_{n-1}\}$, x is adjacent to at most two vertices in $\{w_1, \ldots, w_{n-1}\}$.

Suppose that x is adjacent to at least three vertices of $\{w_1, \ldots, w_{n-1}\}$. Let $i, j \in \{1, \ldots, n-1\}$ with $i \leq j-2$ such that w_i, w_j are neighbors of x. Let $v_k = w_i$ and $v_l = w_j$. Then $\{v_k, x_k, y_k, v_l, x_l, y_l, x\}$ is a cut set of G, a contradiction.

Now, consider $J' = J - \{w_1, \ldots, w_{n-1}\}$. Since J is a plane graph, J' is a plane graph. Let $x \in V(J')$. If $x \in \{w_0, w_n\}$, then by (b) and (c) (since $w_0 = v_0$ and $w_n = v_m$), the degree of x in J' is at least 3. If $x \notin \{w_0, w_n\}$, then by (c), the degree of x in J' is at least 6. Thus the number of edges in J' is at least $\frac{6(|V(J')|-2)+2\cdot 3}{2} = 3|V(J')|-3$, contradicting the planarity of J'. $\square$

Next, we show that (1.4) is best possible by constructing 7-connected obstructions. A *near triangulation* is a plane graph in which every face, with the possible exception of the infinite face, is bounded by a triangle.

First, we construct a near triangulation J whose outer cycle C contains an edge b_1b_2 (with $E(b_1Cb_2) = \{b_1b_2\}$) such that

(1) $d(x) \geq 3$ for all $x \in V(J)$,

(2) $d(x) \geq 7$ for all $x \in V(J - C) \cup \{b_1, b_2\}$,

(3) if T is a cut set of J with $|T| \leq 6$, then $|T| \geq 3$ and every component of $J - T$ contains a vertex of C, and

(4) for each $x \in V(C) - \{b_1, b_2\}$, there does not exist $T \subseteq V(J)$ with $|T| \leq 4$ such that every path from b_2Cx to xCb_1 in $J - b_1b_2$ intersects T.

We start with a path $J_0 = P_0$ consisting of a single edge b_1b_2. Suppose we have constructed a near triangulation J_i and a path P_i in J_i for some $i \geq 0$ such that $P_0 \cap P_i = \emptyset$ and $P_0 \cup P_i$ is contained in the outer cycle of J_i. We construct a near triangulation L_{i+1} as follows. (See Figure 2 for an illustration with $i = 3$.)

Let P_{i+1} be a sufficiently long path (for example, at least 6 times as long as P_i). In the disjoint union of P_{i+1} and L_i, we add edges from each vertex of P_i to at least six consecutive vertices on P_{i+1} so that $P_{i+1} \cup P_0$ is contained in the outer cycle of J_{i+1}, $d(x) \geq 7$ if $x \in V(J_i)$, $d(x) \geq 3$ if $x \in V(P_{i+1})$ and x is not an end of P_{i+1}, and $d(x) = 2$ if x is an end of P_{i+1}.

Let x and x' be the ends of P_4. Add a sufficiently long path Q from x to x' such that Q is internally disjoint from J_4. We add edges from each vertex of $P_4 - \{x, x'\}$ to at least six consecutive vertices on Q such that the result is a near triangulation J, $Q \cup P_0$ is contained in its outer cycle, and $d(x) \geq 7$ if $x \in V(J_4)$ and $d(x) \geq 3$ if $x \in V(Q)$. It is straightforward to check that the resulting graph J satisfying (1)–(4) above. We relabel the vertices in J so that $C = w_0w_1 \dots w_nw_0$ is the outer cycle of J, where $w_0 = b_2 = b$ and $w_n = b_1 = b'$. See Figure 2. We may choose n to be odd.

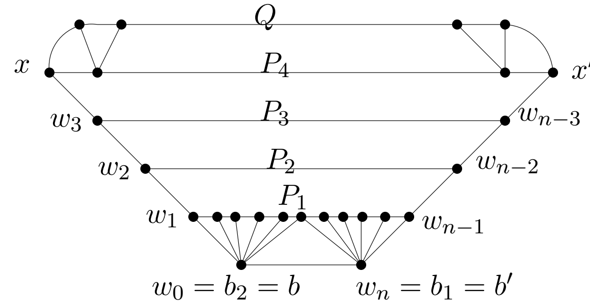

Figure 2: A near triangulation.

Next, we construct a ladder along $v_0 \dots v_m$ whose reduced sequence is $w_0 \dots w_n$, where $m = 2n - 3$ and $n \geq 6$ is an odd integer, and describe how it is attached to J along $w_0 \dots w_n$. See Figure 3.

Let $v_i = w_i$ for $i = 0, 1, 2$, and let $v_{m-2} = w_{n-2}$, $v_{m-1} = w_{n-1}$, and $v_m = w_n$. For $i = 1, 2, m-1, m$, let $x_{i-1} = x_i$ and $y_{i-1} = y_i$, and let R_i' be the complete graph on $\{x_i, y_i, v_{i-1}, v_i\}$. For each $i = 2k+1$, $3 \leq i \leq m-2$, let $x_{i-1} \neq x_i$, $y_{i-1} \neq y_i$, and $v_{i-1} = v_i = w_{k+1}$, and let R_i' be the complete graph on $\{x_{i-1}, x_i, v_i, y_{i-1}, y_i\}$. For each $i = 2k+2$ with $4 \leq i \leq m-3$, let $x_{i-1} = x_i$, $y_{i-1} = y_i$, $v_{i-1} = w_{k+1}$ and $v_i = w_{k+2}$, and let R_i' be the complete graph on $\{x_i, v_{i-1}, v_i, y_i\}$. Let S be the set of 2-element subsets of $\{x_i, v_i, y_i\}$ for all $i = 0, \dots, m$. Let $R_i = R_i' - S$. Then $(R_i, (x_{i-1}, v_{i-1}, y_{i-1}), (x_i, v_i, y_i))$ is a rung. Let $L = (\bigcup_{i=1}^m R_i) + S$. Then $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a ladder.

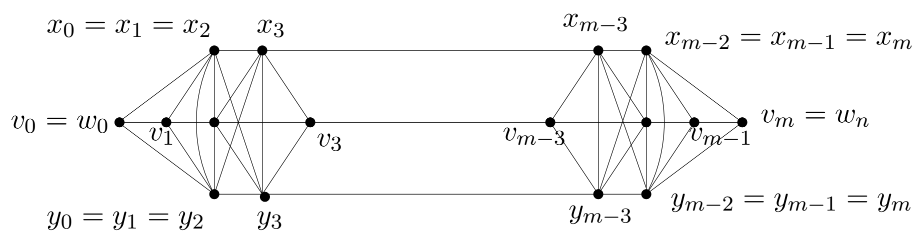

Figure 3: The ladder L.

Finally, let G be obtained from the disjoint union of J and L by identifying the vertices $w_0, \dots, w_n$ in both J and L. Now (J, L) is a separation in G, $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is 3-planar, and $(L, (x_0, v_0, y_0), (x_m, v_m, y_m))$ is a ladder along $v_0v_1 \dots v_m$, where $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.

By (1.3), $(G, \{x_0, y_0\}, \{x_m, y_m\}, (v_0, v_m))$ is an obstruction.

We claim that G is 7-connected. Suppose on the contrary that G is not 7-connected. Then G contains a cut set T with $|T| \leq 6$. Let X be a component of $G - T$. If $V(X) \cap \{w_0, \dots, w_n\} = \emptyset$, then $X \subseteq L - \{w_0, \dots, w_n\}$ (since $|T| \leq 6$ and by (3)), and so, $|T| \geq 7$ (by the construction of L), a contradiction. So $V(X) \cap \{w_0, \dots, w_n\} \neq \emptyset$.

If $(V(X) \cap V(L)) - \{w_0, \dots, w_n\} \neq \emptyset$, then $|T \cap V(L)| \geq 7$ (by the construction of L), a contradiction.

Hence, $V(X) \cap V(L) \subseteq \{w_0, \dots, w_n\}$, and so, X is a component of $J - (T \cap V(J))$ and the neighbors of $V(X) \cap \{w_0, \dots, w_n\}$ in L must be contained in T. By (3), $|T \cap V(J)| \geq 3$. If $V(X) \cap \{w_2, \dots, w_{n-2}\} \neq \emptyset$, then $|(T \cap L) - \{w_0, \dots, w_n\}| \geq 4$, and so, $|T \cap V(J)| \leq 2$, a contradiction. So assume that $V(X) \cap V(L) \subseteq \{w_0, w_1, w_{n-1}, w_n\}$. Furthermore, $V(X) \cap V(L) \subseteq \{w_0, w_1\}$ or $V(X) \cap V(L) \subseteq \{w_{n-1}, w_n\}$; as otherwise, $|(T \cap V(L)) - \{w_0, \dots, w_n\}| \geq 4$, and so, $|T \cap V(J)| \leq 2$, a contradiction. So there is some $x \in V(b_2Cb_1)$ such that every path from b_2Cx to xCb_1 in $J - b_1b_2$ intersects $T \cap V(J)$. Since $|(T \cap V(L)) - \{w_0, \dots, w_n\}| \geq 2$, and so, $|T \cap V(J)| \leq 4$, contradicting (4). □

Seymour pointed out that (1.3) can be used to solve the following problem: Let G be a graph and $\{b, b', x, y\} \subseteq V(G)$; determine if G contains a path from b to b' and through x and y in that order.

To see this, we consider two cases. If $xy \in E(G)$, then this problem reduces to the problem for finding a $(\{b, x\}, \{b', y\})$-linkage in G, and so, can be solved using (2.1). So assume that $xy \notin E(G)$. Let G' be the graph obtained from G by replacing x with vertices a', c' and joining a', c' to all neighbors of x, and replacing y with vertices a, c and joining a, c to all neighbors of y. It is easy to verify that G contains a path from b to b' through x and y in that order iff G' contains three disjoint paths from $\{a, b, c\}$ to $\{a', b', c'\}$ with no path from b to b'.

Another problem mentioned to me by Seymour is to characterize the connected graphs containing cycles through four given vertices in a prescribed order. Using a similar technique as above, this problem can be formulated as follows. Let G be a graph, $\{a_1, a_2, b_1, b_2\} \subseteq V(G)$, and $\{c_1, c_2, d_1, d_2\} \subseteq V(G)$. Characterize those G such that, for any four disjoint paths from $\{a_1, a_2, b_1, b_2\}$ to $\{c_1, c_2, d_1, d_2\}$, the two paths with an end in $\{a_1, a_2\}$ have their other ends both in $\{c_1, c_2\}$ or both in $\{d_1, d_2\}$. The planar case is solved in ([4], Theorem 2.1).

Acknowledgment

I would like to thank Robin Thomas for bringing this problem to my attention. I also would like to thank Paul Seymour for pointing out the connection between the main result of this paper and the problems about paths and cycles through vertices in a specific order.

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