On a bipartition problem of Bollobás and Scott

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Abstract

The bipartite density of a graph G is $\max\{|E(H)|/|E(G)| : H$ is a bipartite subgraph of $G\}$. It is NP-hard to determine the bipartite density of any triangle-free cubic graph. A *biased maximum bipartite subgraph* of a graph G is a bipartite subgraph of G with the maximum number of edges such that one of its partite sets is independent in G. Let $\mathcal{H}$ denote the collection of all connected cubic graphs which have bipartite density $\frac{4}{5}$ and contain biased maximum bipartite subgraphs. Bollobás and Scott asked which cubic graphs belong to $\mathcal{H}$. This same problem was also proposed by Malle in 1982. We show that any graph in $\mathcal{H}$ can be reduced, through a sequence of three types of operations, to a member of a well characterized class. As a consequence, we give an algorithm that decides whether a given graph G belongs to $\mathcal{H}$. Our algorithm runs in polynomial time, provided that G has a constant number of triangles that are not blocks of G and do not share edges with any other triangles in G.

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1 Introduction

It is a well known fact that every graph with m edges contains a bipartite subgraph with at least $\frac{m}{2}$ edges. Edwards [5,6] improved this lower bound to $\frac{m}{2}+\sqrt{\frac{m}{8}+\frac{1}{64}}-\frac{1}{8}$. This bound is essentially best possible as evidenced by the complete graphs K_{2n+1}. The *Maximum Bipartite Subgraph Problem* on a graph G is the problem of finding a *maximum bipartite subgraph* of G, that is, a bipartite subgraph of G with the maximum number of edges; and it is shown in [11] to be NP-hard even for triangle-free cubic graphs.

The Maximum Bipartite Subgraph Problem can also be formulated as follows: Given a graph G, find a partition V_1, V_2 of $V(G)$ that minimizes $\varepsilon(V_1)+\varepsilon(V_2)$, where $\varepsilon(V_i)$, $i \in \{1,2\}$, denotes the number of edges of G with both ends in V_i. This is a type of *judicious partition problems* studied by Bollobás and Scott in [1]: Given a graph G, find a partition $V_1, \ldots, V_k$ of $V(G)$ such that (for some $1 \le t \le k$) all $\{V_{i_1}, \ldots, V_{i_t}\}$ satisfy certain constraints.

In particular, Bollobás and Scott [2] considered the following problem: Given a graph G, find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{\varepsilon(V_1), \varepsilon(V_2)\}$. (This problem in general is NP-hard as shown by Shahrokhi and Székely [9].) They proved that for any graph G with m edges, there is a partition V_1, V_2 of $V(G)$ such that $\varepsilon(V_i) \le \frac{m}{4}+\sqrt{\frac{m}{32}+\frac{1}{256}}-\frac{1}{16}$ (for $i \in \{1,2\}$) and $\varepsilon(V_1, V_2) \ge \frac{m}{2}+\sqrt{\frac{m}{8}+\frac{1}{64}}-\frac{1}{8}$ (the Edwards bound), where $\varepsilon(V_1, V_2)$ denotes the number of edges with one end in V_1 and the other in V_2. Moreover, the complete graphs K_{2n+1} are the only extremal graphs (modulo isolated vertices).

For triangle-free cubic graphs, the Edwards bound above has been improved. For convenience, we follow [4] to define the *bipartite density* of a graph G as

$$b(G) := \max\{\varepsilon(H)/\varepsilon(G) : \ H \text{ is a bipartite subgraph of } G\},$$

where $\varepsilon(H) = |E(H)|$ and $\varepsilon(G) = |E(G)|$. Then the Edwards bound implies $b(G) > \frac{1}{2}$. Hopkins and Stanton [7] proved that $b(G) \ge \frac{4}{5}$ if G is triangle-free and cubic. Bondy and Locke [4] extended the Hopkins-Stanton result to all subcubic graphs; and they showed that the Petersen graph and the dodecahedron are the only connected triangle-free cubic graphs that have bipartite density $\frac{4}{5}$. A graph is said to be *subcubic* if it has maximum degree at most 3.

The subcubic graphs F_i, $1 \le i \le 7$, in Figure 1 are triangle-free and have bipartite density $\frac{4}{5}$. Notice that F_6 is the Petersen graph, and F_7 is the dodecahedron. The vertices of each F_i in Figure 1 are represented by solid circles and solid squares; and the edges joining a solid circle and a solid square induce a maximum bipartite subgraph of F_i. Also note that the solid squares form an independent set in F_i. Recently, the present authors [10] proved the following result, which establishes a conjecture of Bondy and Locke [4].

Theorem 1.1 *(Xu and Yu [10]). The graphs F_i, $1 \le i \le 7$, are precisely those connected triangle-free subcubic graphs that have bipartite density $\frac{4}{5}$.*

For ease of presentation, we introduce some notation. For any bipartite graph B, we use $V_1(B)$ and $V_2(B)$ to denote a partition of $V(B)$ such that every edge of B has exactly one end in each $V_i(B)$, $i \in \{1,2\}$; and such a partition is called a *bipartition* of B, and $V_1(B)$ and $V_2(B)$ are called *partite* sets of B. Given a graph G, let

$$\mathcal{B}(G) = \{\text{spanning maximum bipartite subgraphs of } G\}.$$

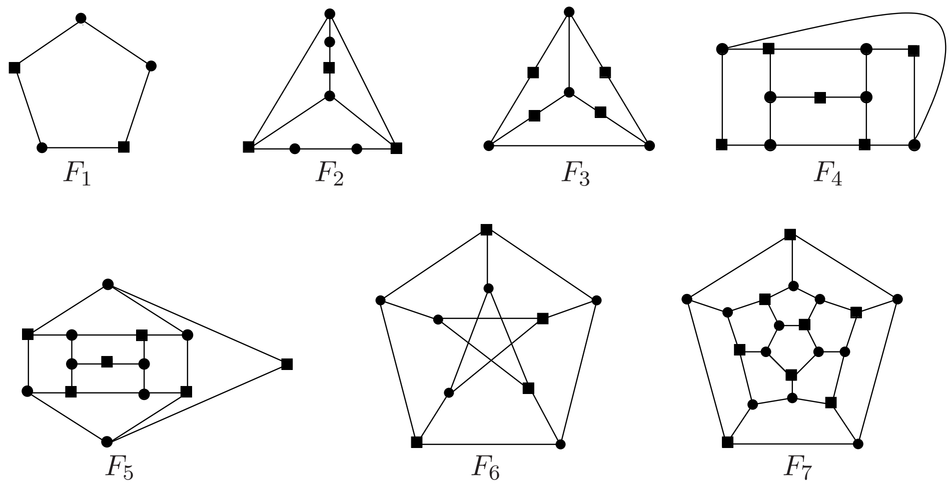

Figure 1: The connected triangle-free subcubic graphs with bipartite density $\frac{4}{5}$.

Note that any maximum bipartite subgraph of G is contained in a member of $\mathcal{B}(G)$ with the same edge set. A member $B \in \mathcal{B}(G)$ is said to be *biased* if there exists a bipartition $V_1(B)$ and $V_2(B)$ of B such that for some $i \in \{1, 2\}$, $V_i(B)$ is independent in G. A biased bipartition of $V(G)$ could be some way from being judicious. Let

$$\mathcal{H} = \left\{ \begin{array}{l} \text{connected subcubic graphs with bipartite density } \frac{4}{5} \\ \text{and containing biased maximum bipartite subgraphs} \end{array} \right\}.$$

Bollobás and Scott [3] observed that the Petersen graph belongs to $\mathcal{H}$, and further asked (Problem 3 in [3]) which cubic graphs are members of $\mathcal{H}$. This problem was also proposed by Malle [8] 1982, in a different form. In [8], Malle define an “elementary path” in a graph as a nontrivial path whose ends have degree at least 3 and whose internal vertices have degree 2. Malle [8] proved that if B is a maximum bipartite subgraph of a cubic graph G, then any elementary path has length at most 5. Malle then defined *abc-cubic* graphs as those cubic graphs which admit a maximum bipartite subgraph B such that each vertex of degree 3 in B is the common end of three elementary paths of length a, b, c, respectively. It is clear that the 111-cubic graphs are precisely the bipartite cubic graphs. Malle showed that there are precisely two triangle-free connected 222-cubic graphs: the Peterson graph and the dodecahedron. Malle [8] then raised the problem of characterizing all 222-cubic graphs. A simple calculation shows that a graph is 222-cubic graph iff it is $\mathcal{H}$.

Note that the graphs $F_i, 1 \leq i \leq 7$, in Figure 1 all contain biased maximum bipartite subgraphs. The following consequence of Theorem 1.1 answers this Bollobás–Scott question for triangle-free graphs.

Corollary 1.2 *(Xu and Yu [10]). If G is a triangle-free subcubic graph and $b(G) = \frac{4}{5}$, then $G \in \mathcal{H}$.*

The purpose of this paper is to solve the above mentioned problem of Bollobás and Scott, by giving a structural description of those cubic graphs that belong to $\mathcal{H}$. We shall show that if $G \in \mathcal{H}$ and G is cubic then, by repeatedly performing three types of operations (starting with G

and a biased maximum bipartite subgraph of G) until no such operation is possible, we obtain an "irreducible" subcubic graph $H \in \mathcal{H}$ and a special biased maximum bipartite subgraph of H. Such irreducible graphs will be characterized completely. As a consequence, any cubic graph in $\mathcal{H}$ may be obtained from an irreducible one by repeatedly applying a sequences of three types of operations, thereby answering the Bollobás-Scott question.

Our structural results can be used to recognize whether a cubic graph belongs to $\mathcal{H}$. Recall that a block of a graph either is induced by a cut edge, or is a maximal 2-connected subgraph. Given a graph G, let $t(G)$ denote the number of triangles in a G that are not blocks of G and do not share edges with any other triangles in G.

Theorem 1.3 *Given any connected cubic graph G, we can decide in $O(|V(G)|^{t(G)+1})$ time whether $G \in \mathcal{H}$.*

Clearly, such algorithm runs in polynomial time if $t(G)$ is a constant or not considered as part of the input. The exponent $t(G)$ is due to the fact that one of the three operations we use is not "reversible". Avoiding the dependency (of running time) on $t(G)$ seems unlikely, since that might require an algorithm that decides, in polynomial time, the bipartite density of a triangle-free subcubic graph (but the Maximum Bipartite Subgraph Problem is NP-hard for triangle-free cubic graphs). It will be clear that our algorithm can be modified so that it finds in $O(|V(G)|^{t(G)+1})$ time a biased maximum bipartite subgraph of G in the case when G is cubic and $G \in \mathcal{H}$.

Precise descriptions of the above-mentioned operations are given in section 2, where we also show how these operations affect membership in $\mathcal{H}$. In section 3, we define and characterize the irreducible graphs. In section 4, we prove Theorem 1.3.

For convenience, we use $A := B$ to rename B to A. Let G be a graph and $S \subseteq V(G) \cup E(G)$. Then $G - S$ denotes the graph obtained from G by deleting S and edges of G incident with vertices in S. For any subgraph H of G, we use $H + S$ to denote the subgraph of G with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{uv \in S \cap E(G) : \{u, v\} \subseteq V(H) \cup (S \cap V(G))\}$. When $S = \{s\}$, we simply write $G - s$ and $H + s$ instead of $G - \{s\}$ and $H + \{s\}$, respectively. For vertices $v_1, \ldots, v_k$ of G, we use $A(v_1, \ldots, v_k)$ to denote the set consisting of $v_i, 1 \le i \le k$, and all edges of G with at least one end in $\{v_1, \ldots, v_k\}$. A vertex of G is said to be a *k-vertex* if it has degree k in G. For any vertex v of G, we use $d(v)$ or $d_G(v)$ to denote the degree of v in G, and use $N(v)$ or $N_G(v)$ to denote the set of neighbors of v in G. For any subgraph H of G, let $N(H)$ or $N_G(H)$ denote $(\bigcup_{v \in V(H)} N(v)) - V(H)$.

As usual, K_n denotes the complete graph on n vertices. We use K_3^+ to denote the graph obtained from K_3 by adding a multiple edge, and K_4^- to denote the graph obtained from K_4 by deleting an edge.

2 Three operations

The graphs in the Bollobás-Scott problem are cubic; but we need to deal with subcubic graphs as well. Recall that Corollary 1.2 characterizes those triangle-free subcubic graphs that are in $\mathcal{H}$. We now describe three operations that may be used to reduce graphs in $\mathcal{H}$ to smaller graphs that are also in $\mathcal{H}$. These operations are performed on triangles and, in most cases, reduce the number of triangles.

Operation I. Let G be a subcubic graph and let S be a subgraph of G isomorphic to K_4^-. Operation I simply contracts S in G; and we use G/S to denote the resulting graph. See Figure 2, where $V(S) = \{x_1, x_2, x_3, x_4\}$ and S is contracted to the vertex u.

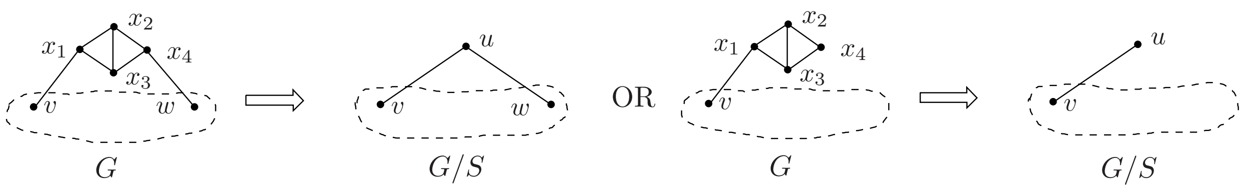

Figure 2: Operation I, contracting a K_4^-.

Note that G/S is subcubic; G/S is connected iff G is connected; and if $|V(G)| \geq 5$ and G is connected, then the vertex in G/S resulted from the contraction of S is a 1-vertex (when $|N_G(S)| = 1$) or a 2-vertex (when $|N_G(S)| = 2$). The following lemma shows that $G \in \mathcal{H}$ iff $G/S \in \mathcal{H}$.

Lemma 2.1 *Let G be a connected subcubic graph with at least five vertices, let S be a subgraph of G isomorphic to K_4^-, and let u denote the vertex of G/S resulted from the contraction of S. Then*

(i) $b(G) = \frac{4}{5}$ iff $b(G/S) = \frac{4}{5}$, and

(ii) G contains a biased maximum bipartite subgraph iff there exists $B' \in \mathcal{B}(G/S)$ such that $\{ux : x \in N_G(S)\} \subseteq E(B')$, $V_1(B')$ is independent in G/S, and $u \in V_1(B')$.

Proof. Let $V(S) = \{x_1, x_2, x_3, x_4\}$ such that $x_1x_4 \notin E(S)$. Let $A := A(x_1, x_2, x_3, x_4)$. Since $|V(G)| \geq 5$ and G is connected, at least one of x_1 and x_4 has degree 3. Let $v \in N_G(x_1) - V(S)$ if $d(x_1) = 3$, and $w \in N_G(x_4) - V(S)$ if $d(x_4) = 3$. See Figure 2. Define $A' := \{ux : x \in N_G(S)\}$. Without loss of generality, we may assume $A' = \{uv\}$ when $|N_G(S)| = 1$.

To prove (i), let $B_1 \in \mathcal{B}(G)$ and $B_2' \in \mathcal{B}(G/S)$. Define

$$B_1' := \begin{cases} (B_1 - V(S)) + (A' \cup \{u\}), & \text{if } N_G(S) \subseteq V_i(B_1) \text{ for some } i \in \{1, 2\}; \\ (B_1 - V(S)) + \{u, uv\}, & \text{otherwise.} \end{cases}$$

Then B_1' is a bipartite subgraph of G/S. Since $B_1 \in \mathcal{B}(G)$, B_1 uses 6 edges from A when $|N_G(S)| = 2$ and $N_G(S) \subseteq V_i(B)$ for some $i \in \{1, 2\}$, and B_1 uses five edges from A otherwise. Hence, $\varepsilon(B_1') = \varepsilon(B_1) - 4 \leq \varepsilon(B_2')$ (since $B_2' \in \mathcal{B}(G/S)$). Also define

$$B_2 := \begin{cases} (B_2' - u) + (A - \{x_2x_3\}), & \text{if } N_G(S) \subseteq V_i(B_2') \text{ for some } i \in \{1, 2\}; \\ (B_2' - u) + (A - \{x_1x_2, x_3x_4\}), & \text{otherwise.} \end{cases}$$

Then B_2 is a bipartite subgraph of G. Since $B_2' \in \mathcal{B}(G/S)$, $A' \subseteq E(B_2')$ when $N_G(S) \subseteq V_i(B_2')$ for some $i \in \{1, 2\}$, and $|A' \cap E(B_2')| = 1$ otherwise. So $\varepsilon(B_2) = \varepsilon(B_2') + 4 \leq \varepsilon(B_1)$ (since $B_1 \in \mathcal{B}(G)$).

Suppose $b(G) = \frac{4}{5}$. Then $\varepsilon(B_1) = \frac{4}{5}\varepsilon(G)$, and hence $\varepsilon(B_1') = \frac{4}{5}\varepsilon(G) - 4 = \frac{4}{5}\varepsilon(G/S)$. So $b(G/S) \geq \frac{4}{5}$. Moreover, $b(G/S) = \frac{4}{5}$. For otherwise, $\varepsilon(B_2') > \frac{4}{5}\varepsilon(G/S)$. Hence $\varepsilon(B_2) > \frac{4}{5}\varepsilon(G/S) + 4 = \frac{4}{5}\varepsilon(G)$, which implies $b(G) > \frac{4}{5}$, a contradiction.

Now suppose $b(G/S) = \frac{4}{5}$. Then $\varepsilon(B_2') = \frac{4}{5}\varepsilon(G/S)$, and hence $\varepsilon(B_2) = \frac{4}{5}\varepsilon(G/S) + 4 = \frac{4}{5}\varepsilon(G)$. So $b(G) \geq \frac{4}{5}$. Indeed $b(G) = \frac{4}{5}$. For otherwise, $\varepsilon(B_1) > \frac{4}{5}\varepsilon(G)$. Hence $\varepsilon(B_1') > \frac{4}{5}\varepsilon(G) - 4 = \frac{4}{5}\varepsilon(G/S)$, which implies $b(G/S) > \frac{4}{5}$, a contradiction. This proves (i).

To prove (ii), we first note that every maximum bipartite subgraph of G contains five or six edges from A.

Suppose $B' \in \mathcal{B}(G/S)$ such that $A' \subseteq E(B')$, $V_1(B')$ is independent in G/S, and $u \in V_1(B')$. Then $N_G(S) \subseteq V_2(B')$, and so $B := (B' - u) + (A - \{x_2x_3\})$ is a bipartite subgraph of G, $N_G(S) \subseteq V_2(B)$, and $V_1(B)$ is independent in G. It suffices to show $B \in \mathcal{B}(G)$. Clearly, B uses $4 + |A'|$ edges from A. Suppose $B \notin \mathcal{B}(G)$; then for any $B_1 \in \mathcal{B}(G)$, $\varepsilon(B_1) > \varepsilon(B)$. Since B_1 uses at most $4 + |A'|$ edges from A while B uses $4 + |A'|$ edges from A, we must have $\varepsilon(B_1 - V(S)) > \varepsilon(B - V(S))$. Therefore, $N_G(S) \not\subseteq V_i(B_1)$ for any $i \in \{1, 2\}$; as otherwise, $(B_1 - V(S)) + (A' \cup \{u\})$ is a bipartite subgraph of G/S with more edges than B', a contradiction. However, this shows that $|A'| = 2$ and B_1 uses exactly 5 edges from A. So $\varepsilon(B_1 - V(S)) \geq \varepsilon(B - V(S)) + 2$. Hence $(B_1 - V(S)) + \{u, uv\}$ is a bipartite subgraph of G/S with more edges than B', a contradiction.

Now suppose $B \in \mathcal{B}(G)$ and $V_1(B)$ is independent in G. We claim that $\{x_1, x_4\} \subseteq V_1(B)$. If $\{x_1, x_4\} \subseteq V_2(B)$, then $\{x_2, x_3\} \subseteq V_1(B)$ (by the maximality of B), contradicting the independence of $V_1(B)$. Now suppose $\{x_1, x_4\} \not\subseteq V_i(B)$ for any $i \in \{1, 2\}$. Then by symmetry, we may assume $x_1 \in V_1(B)$ and $x_4 \in V_2(B)$. Since $V_1(B)$ is independent in G, $\{x_2, x_3\} \subseteq V_2(B)$. This implies that B uses at most 4 edges from A, a contradiction (since B uses at least 5 edges from A). Therefore, $\{x_1, x_4\} \subseteq V_1(B)$.

Since $V_1(B)$ is independent in G, $N_G(S) \subseteq V_2(B)$; and hence $B' := (B - V(S)) + (A' \cup \{u\})$ is a bipartite subgraph of G/S, $u \in V_1(B')$, and $V_1(B')$ is independent in G/S. Clearly, B' uses $4 + |A'|$ edges from A. It remains to prove $B' \in \mathcal{B}(G/S)$. Otherwise, suppose $B' \notin \mathcal{B}(G/S)$. Then for any $B_1' \in \mathcal{B}(G/S)$, $\varepsilon(B_1') > \varepsilon(B')$. If $N_G(S) \subseteq V_i(B_1')$ for some $i \in \{1, 2\}$ then $B_1 := (B_1' - u) + (A - \{x_2x_3\})$ is a bipartite subgraph of G with more edges than B, a contradiction. So $N_G(S) \not\subseteq V_i(B_1')$ for any $i \in \{1, 2\}$. Then $|A'| = 2$ and B_1 uses 5 edges from A (while B uses 6 edges from A). So $\varepsilon(B_1' - u) \geq \varepsilon(B' - u) + 2$. Hence, $(B_1' - u) + (A - \{x_1x_2, x_3x_4\})$ is a bipartite subgraph of G with more edges than B, a contradiction. ∎

To describe the other two operations, let us define an *i-triangle* in a graph G to be any triangle in G which contains exactly i 2-vertices of G.

Operation II. Let T be a 1-triangle in a subcubic graph G. Operation II deletes T from G, and the resulting graph is $G - V(T)$. See Figure 3, where $V(T) = \{w, x, y\}$ and $N_G(T) = \{u, v\}$.

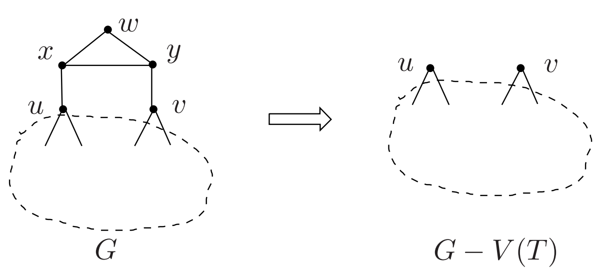

Figure 3: Operation II, removing a 1-triangle.

Clearly $G - V(T)$ is subcubic. Moreover, if T is not a block of G and G is connected, then $G - V(T)$ is connected. The next lemma implies that if $G \in \mathcal{H}$ and T is not a block of G then $G - V(T) \in \mathcal{H}$.

Lemma 2.2 *Let G be a subcubic graph, let T be a 1-triangle in G, and let $H := G - V(T)$. Then* *(i)* $b(G) = \frac{4}{5}$ *iff* $b(H) = \frac{4}{5}$, *and*

(ii) if $b(G) = \frac{4}{5}$ *and there exists* $B \in \mathcal{B}(G)$ *such that* $V_1(B)$ *is independent in* G*, then* $B \cap H \in \mathcal{B}(H)$*,* $V_1(B \cap H)$ *is independent, and*

$$|\{t \in V_2(B \cap H) : d_H(t) \leq 2\}| \leq |\{t \in V_2(B) : d_G(t) \leq 2\}|.$$

Proof. Let $V(T) = \{w, x, y\}$ with $d_G(w) = 2$, and let $u \in N_G(x) - V(T)$ and $v \in N_G(y) - V(T)$. See Figure 3. Let $A := A(w, x, y)$. Then, every bipartite graph in $\mathcal{B}(G)$ contains at most four edges from A. Indeed, for any $B \in \mathcal{B}(G)$, B uses exactly four edges from A, because $(B \cap H) + (A - \{xy\})$ (when $\{u, v\} \subseteq V_i(B)$ for some $i \in \{1, 2\}$) or $(B \cap H) + (A - \{wx\})$ (when $\{u, v\} \not\subseteq V_i(B)$ for any $i \in \{1, 2\}$) is a bipartite subgraph of G.

To prove (i), let $B_1 \in \mathcal{B}(G)$ and $B_2' \in \mathcal{B}(H)$. Then $B_1 \cap H$ is a bipartite subgraph of H. So $\varepsilon(B_2') \geq \varepsilon(B_1 \cap H) = \varepsilon(B_1) - 4$. Define

$$B_2 := \begin{cases} B_2' + (A - \{xy\}), & \text{if } \{u, v\} \subseteq V_i(B_2') \text{ for some } i \in \{1, 2\}; \\ B_2' + (A - \{wx\}), & \text{otherwise.} \end{cases}$$

Then B_2 is a bipartite subgraph of G, and $\varepsilon(B_2) = \varepsilon(B_2') + 4$.

Suppose $b(G) = \frac{4}{5}$. Then $\varepsilon(B_1) = \frac{4}{5}\varepsilon(G)$. Hence, $\varepsilon(B_2') \geq \varepsilon(B_1) - 4 = \frac{4}{5}\varepsilon(G) - 4 = \frac{4}{5}\varepsilon(H)$. Moreover, if $\varepsilon(B_2') > \frac{4}{5}\varepsilon(H)$ then $\varepsilon(B_2) = \varepsilon(B_2') + 4 > \frac{4}{5}\varepsilon(H) + 4 = \frac{4}{5}\varepsilon(G)$, contradicting $b(G) = \frac{4}{5}$. So $\varepsilon(B_2') = \frac{4}{5}\varepsilon(H)$. Hence, $b(H) = \frac{4}{5}$.

Now suppose $b(H) = \frac{4}{5}$. Then $\varepsilon(B_2') = \frac{4}{5}\varepsilon(H)$. So $\varepsilon(B_2) = \varepsilon(B_2') + 4 = \frac{4}{5}\varepsilon(G)$. Hence, $\varepsilon(B_1) \geq \frac{4}{5}\varepsilon(G)$. If $\varepsilon(B_1) > \frac{4}{5}\varepsilon(G)$, then $\varepsilon(B_2') \geq \varepsilon(B_1 \cap H) > \frac{4}{5}\varepsilon(G) - 4 = \frac{4}{5}\varepsilon(H)$, a contradiction. So $\varepsilon(B_1) = \frac{4}{5}\varepsilon(G)$. Hence, $b(G) = \frac{4}{5}$. This proves (i).

Next, we prove (ii). Suppose $b(G) = \frac{4}{5}$, $B \in \mathcal{B}(G)$, and $V_1(B)$ is independent in G. Then $b(H) = \frac{4}{5}$ (by (i)), and hence, $B \cap H \in \mathcal{B}(H)$ (since $\varepsilon(B \cap H) = \frac{4}{5}\varepsilon(H)$). Note that $V_1(B \cap H) = V_1(B) \cap V(H)$ is independent in H.

If $w \in V_1(B)$, then $\{x, y\} \subseteq V_2(B)$ (since $V_1(B)$ is independent) and $\{u, v\} \subseteq V_1(B)$ (since B uses four edges from A). Thus $\{t \in V_2(B \cap H) : d_H(t) \leq 2\} = \{t \in V_2(B) : d_G(t) \leq 2\}$, and (ii) holds.

Now assume $w \in V_2(B)$. Since $V_1(B)$ is independent and B uses four edges from A, $\{x, y\} \not\subseteq V_i(B)$ for any $i \in \{1, 2\}$. By symmetry, we may assume that $x \in V_1(B)$ and $y \in V_2(B)$. Then, again since $V_1(B)$ is independent and B uses four edges from A, $u \in V_2(B)$ and $v \in V_1(B)$. Since $d_H(u) = d_G(u) - 1$, $\{t \in V_2(B \cap H) : d_H(t) \leq 2\} = (\{t \in V_2(B) : d_G(t) \leq 2\} \cup \{u\}) - \{w\}$ when $d_G(u) = 3$, and $\{t \in V_2(B \cap H) : d_H(t) \leq 2\} = \{t \in V_2(B) : d_G(t) \leq 2\} - \{w\}$ when $d_G(u) = 2$. So (ii) holds. ∎

Operation III. Let T be a 0-triangle in a subcubic graph G, and let $u, v \in N(T)$. Operation III deletes T and adds the edge uv, and the resulting graph is $(G - V(T)) + uv$. See Figure 4, where $V(T) = \{x, y, z\}$ and $N_G(T) = \{u, v, w\}$.

Note that if G is connected and wz is not a cut edge of G, then $(G - V(T)) + uv$ is connected. Next we show that Operation III, when applied appropriately, preserves membership in $\mathcal{H}$.

Lemma 2.3 *Let* $G \in \mathcal{H}$*, and let* T *be a 0-triangle in* G*. Then for any* $B \in \mathcal{B}(G)$ *with* $V_1(B)$ *independent,*

(i) $|N_G(T) \cap V_1(B)| = 2$*, and*

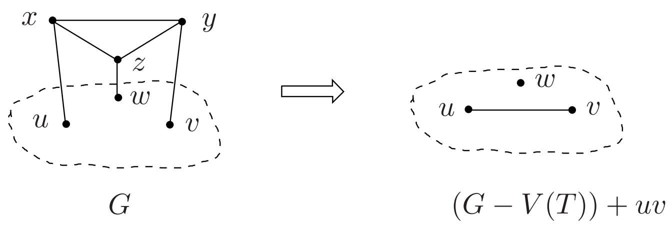

Figure 4: Operation III, removing a 0-triangle and adding an edge.

(ii) if $u, v \in N_G(T)$ and $\{u, v\} \not\subseteq V_1(B)$, then $H := (G - V(T)) + uv \in \mathcal{H}$, $B' := (B - V(T)) + uv \in \mathcal{B}(H)$, $V_1(B')$ is independent in H, and

$$\{t \in V_2(B') : d_H(t) \leq 2\} = \{t \in V_2(B) : d_G(t) \leq 2\}.$$

Proof. Let $V(T) = \{x, y, z\}$, $N_G(T) = \{u, v, w\}$ with $ux, vy, wz \in E(G)$, $A := A(x, y, z)$, and $H := (G - V(T)) + uv$. See Figure 4. Let $B \in \mathcal{B}(G)$ such that $V_1(B)$ is independent.

Note that B contains at least four edges from A, and exactly five edges from A if $|N_G(T) \cap V_i(B)| = 2$ for some $i \in \{1, 2\}$. To see this, we first observe that at least two of $\{u, v, w\}$ are contained in $V_i(B)$ for some $i \in \{1, 2\}$, say u, v. Now $(B - \{x, y, z\}) + (A - \{wz, xy\})$ is a bipartite subgraph of G which contains four edges from A. Now let $|N_G(T) \cap V_i(B)| = 2$ for some $i \in \{1, 2\}$, and by symmetry assume $u \in V_1(B)$ and $\{v, w\} \subseteq V_2(B)$. Then $(B - \{x, y, z\}) + (A - \{yz\})$ is a bipartite subgraph of G which contains five edges from A.

Suppose $|N_G(T) \cap V_1(B)| = 0$. Then $\{u, v, w\} \subseteq V_2(B)$. Since $|\{x, y, z\} \cap V_1(B)| \leq 1$ (because $V_1(B)$ is independent), B uses at most three edges from A, a contradiction. Now assume $|N_G(T) \cap V_1(B)| = 1$. Then $|N_G(T) \cap V_2(B)| = 2$, and we may assume without loss of generality that $u \in V_1(B)$ and $\{v, w\} \subseteq V_2(B)$. Then, since $V_1(B)$ is independent, $x \in V_2(B)$ and at most one of $\{y, z\}$ is in $V_1(B)$. So B uses at most four edges from A, a contradiction. Finally, assume $|N_G(T) \cap V_1(B)| = 3$. Then, $\{x, y, z\} \subseteq V_2(B)$ (since $V_1(B)$ is independent), and B uses just three edges from A, a contradiction. Thus $|N_G(T) \cap V_1(B)| = 2$, and (i) holds.

To prove (ii), we assume by (i) and symmetry that $\{u, w\} \subseteq V_1(B)$ and $v \in V_2(B)$. Then B contains five edges from A. Since $G \in \mathcal{H}$, $\varepsilon(B) = \frac{4}{5}\varepsilon(G)$. Clearly, $B' := (B - A) + uv$ is a bipartite subgraph of H, $V_1(B') = V_1(B \cap H)$ is independent in H, and $\{t \in V_2(B') : d_H(t) \leq 2\} = \{t \in V_2(B) : d_G(t) \leq 2\}$. Since B uses five edges from A, $\varepsilon(B') = \varepsilon(B) - 4 = \frac{4}{5}\varepsilon(H)$. It remains to show that $B' \in \mathcal{B}(H)$. Suppose $B' \notin \mathcal{B}(H)$. Then there exists $B'_1 \in \mathcal{B}(H)$ with $\varepsilon(B'_1) > \varepsilon(B')$. Without loss of generality, assume $u \in V_1(B'_1)$. Then, $v \in V_2(B'_1)$ if $uv \in E(B'_1)$, and $v \in V_1(B'_1)$ if $uv \notin E(B'_1)$ (by the maximality of B'_1). Define

$$B_1 := \begin{cases} (B'_1 - uv) + (A - \{xz\}), & \text{if } uv \in E(B'_1) \text{ and } w \in V_1(B'_1); \\ (B'_1 - uv) + (A - \{yz\}), & \text{if } uv \in E(B'_1) \text{ and } w \in V_2(B'_1); \\ B'_1 + (A - \{xy, wz\}), & \text{if } uv \notin E(B'_1) \text{ and } w \in V_1(B'_1); \\ B'_1 + (A - \{xy\}), & \text{if } uv \notin E(B'_1) \text{ and } w \in V_2(B'_1). \end{cases}$$

Then B_1 is a bipartite subgraph of G. However, $\varepsilon(B_1) \geq \varepsilon(B'_1) + 4 > \varepsilon(B') + 4 = \varepsilon(B)$, a contradiction. ∎

Unlike Operations I and II, we do not have a result for Operation III that is similar to (i) of Lemmas 2.1 and 2.2. That is, Operation III need not preserve bipartite density. For example,

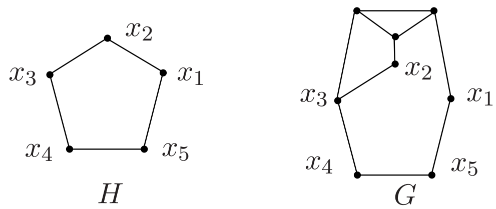

Figure 5: Operation III, decreasing bipartite density.

let H be the cycle $x_1x_2x_3x_4x_5x_1$. Let G be obtained from H by deleting x_1x_2 and by adding a triangle T and a matching from $V(T)$ to $\{x_1, x_2, x_3\}$ (see Figure 5). Then $H = (G - V(T)) + x_1x_2$ and $b(H) = \frac{4}{5}$; but $b(G) > \frac{4}{5}$. (This is the reason why the running time in Theorem 1.3 depends on $t(G)$.) In section 4, we shall give a more detailed analysis of the effect that Operation III has on bipartite density.

3 Irreducible graphs

In this section, we characterize those graphs that can be obtained from cubic graphs in $\mathcal{H}$ by repeatedly applying Operations I, II and III until no such operation is possible.

Let $G \in \mathcal{H}$. We say that G is *irreducible* if (1) no subgraph of G is isomorphic to K_4^-, (2) any 1-triangle in G is a block of G, and (3) if T is a 0-triangle in G then $G[V(T)]$ is a block of G, or $|N(T) \cap V_1(B)| \neq 2$ whenever $B \in \mathcal{B}(G)$ and $V_1(B)$ is independent in G. Here, $G[V(T)]$ denotes the subgraph of G induced by the vertices in $V(T)$.

Lemmas 2.1, 2.2 and 2.3 show that if $G \in \mathcal{H}$ and G is not irreducible, then we may reduce G to a smaller graph in $\mathcal{H}$ by an appropriate application of Operation I, or Operation II, or Operation III. (We only apply Operations II and III on triangles that do not induce blocks.) Therefore, starting with any irreducible graph H and a biased maximum bipartite subgraph B of H, we may apply the "inverse" operations of Operations I, II and III to construct all graphs in $\mathcal{H}$. However, these operations do not give rise to a polynomial time algorithm for deciding membership in $\mathcal{H}$. This is due to the fact (see Figure 5) that Operation III need not preserve bipartite density.

The next result gives a structural description of irreducible graphs.

Theorem 3.1 *Let G be an irreducible graph. Then every block of G is either a triangle, or a K_3^+, or a triangle-free graph.*

Proof. It suffices to show that every triangle in G induces a block of G. Since $G \in \mathcal{H}$, G is connected and $|V(G)| \geq 4$. Let T be a triangle contained in G. If T is a 1-triangle then, by definition of irreducible graphs, T must be a block of G. If T is a 2-triangle, then the only 3-vertex of G in T is a cut vertex of G (since $|V(G)| \geq 4$), and hence, T is a block of G. So we may assume that T is a 0-triangle. We may further assume that $G[V(T)] = T$; for otherwise, since G is subcubic, $G[V(T)]$ is isomorphic to K_3^+, and hence is a block of G. We proceed to show that T is a block of G.

Let $V(T) = \{x, y, z\}$, and let $N(T) = \{u, v, w\}$ such that $xu, yv, zw \in E(G)$. Since G is irreducible, it does not contain a K_4^-. Hence, because $G[V(T)] = T$, $|\{u, v, w\}| = 3$. Let

$A := A(x, y, z)$. Let $B \in \mathcal{B}(G)$ such that $V_1(B)$ is independent. Without loss of generality, we may assume $\{u, v\} \subseteq V_i(B)$ for some $i \in \{1, 2\}$. Then B uses at least 4 edges from A, since $B \in \mathcal{B}(G)$ and $(B - \{x, y, z\}) + (A - \{xy, zw\})$ is a bipartite subgraph of G.

If $\{u, v, w\} \subseteq V_1(B)$ then, since $V_1(B)$ is independent, $\{x, y, z\} \subseteq V_2(B)$, and so, B uses three edges from A, a contradiction. If $\{u, v, w\} \subseteq V_2(B)$ then, since $V_1(B)$ is independent, at most one of $\{x, y, z\}$ belongs to $V_1(B)$, which implies that B uses at most three edges from A, again a contradiction.

Now suppose $\{u, v\} \subseteq V_2(B)$ and $w \in V_1(B)$. Then, since $V_1(B)$ is independent, $z \in V_2(B)$, and $x \in V_2(B)$ or $y \in V_2(B)$. So B uses at most four edges from A. However, $(B - \{x, y, z\}) + (A - \{xy\})$ is a bipartite subgraph of G with more edges than B, a contradiction.

Therefore, $\{u, v\} \subseteq V_1(B)$ and $w \in V_2(B)$. Since G is irreducible, it follows from the definition of irreducible graphs that T is a block of G. ∎

Theorem 3.1 tells us that in order to further understand the structure of irreducible graphs, we need to have better knowledge of 2-connected triangle-free blocks. However, this seems to be difficult, partly due to the presence of 2-triangles. When an irreducible graph contains 2-triangles, its triangle-free blocks are hard to characterize. For instance, such triangle-free blocks can be an arbitrary even cycle, in addition to the graphs in Figure 1. Take the disjoint union of a cycle $x_1 \ldots x_{2n} x_1$ ($n \geq 2$) and triangles $T_i = u_i v_i w_i u_i$ ($1 \leq i \leq 2n$), and add edges $w_i x_i$ ($1 \leq i \leq 2n$). Let G_n denote the resulting graph. See Figure 6 for G_2, G_3 and G_4. Clearly, every maximum bipartite subgraph of G_n can be obtained from G_n by deleting one edge from each T_i. So a simple calculation shows that $b(G_n) = \frac{4}{5}$. Moreover, $B_n := G_n - \{u_{2i-1}v_{2i-1}, u_{2i}w_{2i} \ : \ i = 1, 2, \ldots, n\}$ is a maximum bipartite subgraph of G_n, and $V_1(B_n) = \{v_{2i}, x_{2i}, w_{2i-1} \ : \ i = 1, 2, \ldots, n\}$ is independent in G_n.

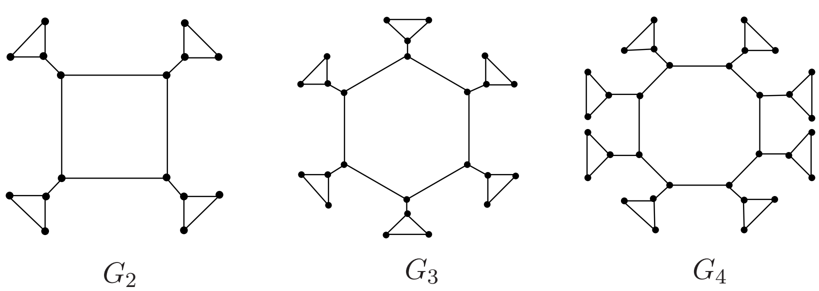

Figure 6: Graphs G_n for $n = 2, 3, 4$.

On the other hand, we can show that 2-triangles may be avoided when reducing a cubic graph in $\mathcal{H}$. As a consequence, the triangle-free blocks of the resulting irreducible graphs can be characterized completely, thereby solving the Bollobás-Scott problem. In order to state our results precisely, we introduce the notion of *fragment*.

Let G be a subcubic graph such that each of its blocks is a triangle, or a K_3^+, or a triangle-free graph. Note that a block of G that is a K_3^+ must be an endblock of G. (An endblock of G is a block of G that contains at most one cut evrtex of G.) Let H denote the subgraph of G induced by those edges of G that are not contained in any 2-connected triangle-free blocks of G and are not incident with a vertex of any K_3^+. Note that H need not be connected, and each block of H is either a triangle or a K_2. See Figure 7 for an example of G and H. Let S denote the set of vertices of H that have degree at least 2 and are not contained in any triangle of H. (In Figure 7, the vertices belonging to S are represented by solid squares.) We define a *fragment* of G as the subgraph of H induced by all edges of a component of $H-S$ and all edges from that component to S. In particular, if K is a fragment of G, then every block of K is a triangle or a K_2, and every endblock of K is a 2-triangle or a K_2. If, in addition, $|V(K)|\geq 3$ then every edge of K is incident with a vertex of some triangle in K.

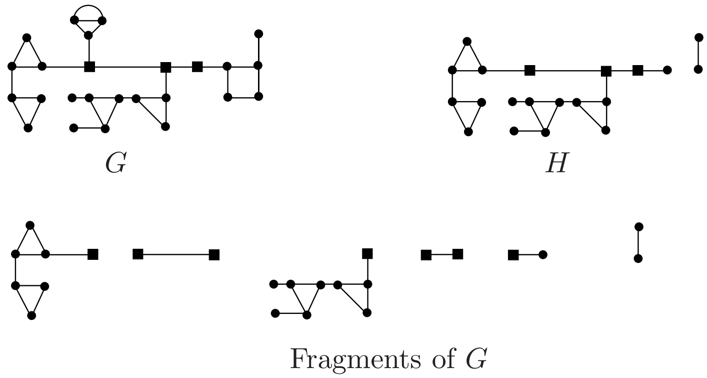

Figure 7: An example of fragments.

We shall show that when Operation III is applied appropriately in the reduction of a cubic graph in $\mathcal{H}$, the fragments of the resulting irreducible graph are actually *simple*, namely, containing no 2-triangles. But first, we determine the bipartite density of a simple fragment.

Lemma 3.2 *Let F be a simple fragment with p triangles, k of which are 1-triangles, where $p\geq k\geq 0$. Then $b(F)=\frac{4p-(k-1)}{5p-(k-1)}$.*

Proof. Since F contains p vertex disjoint triangles, one has to remove at least p edges to make F bipartite. Conversely, since each block of F is a triangle or K_2, we can remove one edge from each triangle to make F bipartite. Therefore, for every $B\in\mathcal{B}(F)$, $\varepsilon(B)=\varepsilon(F)-p$. Thus $b(F)=\frac{\varepsilon(F)-p}{\varepsilon(F)}$.

It suffices to show that $\varepsilon(F)=5p-(k-1)$. Since F is a simple fragment, F has no 2-triangles, and hence every endblock of F is a K_2. When $p=0$, we have $F=K_2$ and $\varepsilon(F)=5p-(k-1)$. Suppose $p=1$. Then $\varepsilon(F)=6$ if $k=0$, and $\varepsilon(F)=5$ if $k=1$. So $\varepsilon(F)=5p-(k-1)$ when $p=1$. Now assume $p\geq 2$. We may choose a triangle T in F so that at most one of its three vertices is contained in a block of F that is neither a triangle nor an endblock. Let $w\in V(T)$ be contained in an endblock of F that is a K_2. Let u,v be the other two vertices in T.

Suppose T is a 1-triangle. Then $k\geq 1$. Without loss of generality, we may assume v is the 2-vertex in F, and let wx be the edge inducing an endblock of F. Then $F':=F-\{v,w,x\}$ is a simple fragment with $p-1$ triangles, $k-1$ of which are 1-triangles. Hence by induction, $\varepsilon(F')=5(p-1)-((k-1)-1)=5(p-1)-(k-1)+1$. So $\varepsilon(F)=\varepsilon(F')+4=5p-(k-1)$.

Now assume that T is 0-triangle. Since $p\geq 2$, we may assume that u is not contained in any endblock of F. By the choice of T, we may let $x,y\in N_F(T)$ such that vx,wy each induce an endblock of F. Then $F':=F-\{v,w,x,y\}$ is a simple fragment with $p-1$ triangles, k of which are 1-triangles. By induction, $\varepsilon(F')=5(p-1)-(k-1)$; and $\varepsilon(F)=\varepsilon(F')+5=5p-(k-1)$. ∎

We now prove that a cubic graph in $\mathcal{H}$ can be reduced to a well characterized class of irreducible graphs.

Theorem 3.3 *Let $G \in \mathcal{H}$ be cubic, and let B be a biased maximum bipartite subgraph of G. Then either $G \in \{F_6, F_7\}$, or we can repeatedly apply Operations I, II and III, starting with G and B, to obtain an irreducible graph H and a bipartite graph $B' \in \mathcal{B}(H)$ such that*

(i) $V_1(B')$ is independent and every vertex in $V_2(B')$ is a 3-vertex of H,

(ii) H contains no 2-triangles,

(iii) every fragment of H is simple and has exactly one 1-triangle, and

(iv) every 2-connected triangle-free block of H belongs to $\{F_i : 3 \le i \le 5\}$.

Proof. Since $G \in \mathcal{H}$, G is connected and $b(G) = \frac{4}{5}$. Since B is biased, we may assume that $V_1(B)$ is independent in G. Since G is a cubic graph, $\{t \in V_2(B) : d_G(t) \le 2\} = \emptyset$. Hence, by repeatedly applying Operations I, II and III, starting with G and B, we obtain (by Lemmas 2.1, 2.2 and 2.3) a connected subcubic graph H and a bipartite graph $B' \in \mathcal{B}(H)$ such that $b(H) = \frac{4}{5}$, H is irreducible, $V_1(B')$ is independent in H, and

$$(*)\ |\{t \in V_2(B') : d_H(t) \le 2\}| \le |\{t \in V_2(B) : d_G(t) \le 2\}| = 0.$$

In particular, $(*)$ follows from (ii) of Lemmas 2.1, 2.2 and 2.3.

By $(*)$, every vertex of $V_2(B')$ is a 3-vertex of H. Therefore, since $V_1(B')$ is independent in H, we have (i). Moreover, H contains no 2-triangles; for otherwise, since $V_1(B')$ is independent, one of the 2-vertices of that 2-triangle must belong to $V_2(B')$, contradicting $(*)$. So (ii) holds.

By Theorem 3.1, every block of H is a triangle, or a K_3^+, or a triangle-free graph. (Moreover, any K_3^+ in H is necessarily an endblock of H.) By (ii), no endblock of H is triangle. Let $G_1, \ldots, G_m$ denote the fragments of H, and let $H_1, \ldots, H_n$ denote the 2-connected triangle-free blocks of H. Let $D_1, \ldots, D_k$ denote the connected subgraphs of H, each of which is induced by edges incident with vertices of some K_3^+. Clearly, each D_l has 4 vertices and 5 edges, and $b(D_l) = \frac{4}{5}$ for $1 \le l \le k$.

We claim that every 2-vertex in G_i (for each $1 \le i \le m$) is contained in $V_1(B')$. Let v be a 2-vertex in G_i. If v is also a 2-vertex in H then, by $(*)$, we have $v \in V_1(B')$. So we may assume that v is not a 2-vertex in H. Then v must be a 1-vertex in some D_l $(1 \le l \le k)$. The maximality of B' implies that B' must use the multiple edges in D_l and that B' uses four edges of D_l. Since $V_1(B')$ is independent in H, $v \in V_1(B')$.

Therefore, since $V_1(B')$ is independent in H, no G_i has a 2-triangle. So $G_1, \ldots, G_m$ are simple fragments. Let k_i be the number of 1-triangles in G_i $(1 \le i \le m)$. We now prove $k_i \in \{0, 1\}$. For otherwise, assume $k_i \ge 2$ for some $i \in \{1, \ldots, m\}$. Then G_i contains a path $x_1y_1x_2y_2\ldots x_sy_s$ such that for each $p \in \{1, \ldots, s\}$, x_py_p belongs to a triangle $T_p := x_py_pz_px_p$, and both T_1 and T_s are 1-triangles. Since each block of G_i is a triangle or a K_2, $\{z_j : 1 \le p \le s\} \cap \{x_p, y_p : 1 \le p \le s\} = \emptyset$ and $z_p \ne z_q$ whenever $1 \le p \ne q \le s$. Without loss of generality, we assume $d_{G_i}(z_1) = d_{G_i}(z_s) = 2$. Since $B' \in \mathcal{B}(H)$, $\{y_{p-1}x_p : \ 2 \le p \le s\} \subseteq E(B')$. Because z_1 is a 2-vertex in G_i, $z_1 \in V_1(B')$. Since $V_1(B')$ is independent in H, $x_1, y_1 \in V_2(B)$. Therefore, because $y_1x_2 \in E(B')$ and $V_1(B')$ is independent, $x_2 \in V_1(B')$ and $y_2, z_2 \in V_2(B')$. Now assume that $x_p \in V_1(B')$ and $y_p, z_p \in V_2(B')$ for some $2 \le p < s$. Then $x_{p+1} \in V_1(B')$ (since $y_px_{p+1} \in E(B')$), and hence, $y_{p+1}, z_{p+1} \in V_2(B')$ (since $V_1(B')$ is independent). So from induction, we conclude that $y_s, z_s \in V_2(B')$. However, z_s is a 2-vertex in G_i, and hence should be in $V_1(B')$, a contradiction.

Since $G_1, \ldots, G_m$ are simple fragments, it follows from Lemma 3.2 that $b(G_i) = \frac{4p-(k_i-1)}{5p-(k_i-1)}$. Clearly, $b(G_i) > \frac{4}{5}$ if $k_i = 0$, and $b(G_i) = \frac{4}{5}$ if $k_i = 1$. By the Bondy-Locke result in [4], $b(H_j) \ge \frac{4}{5}$

for $1 \leq j \leq n$ (since each H_j is subcubic and triangle-free). Therefore, we must have $k_i = 1$ and $b(G_i) = b(H_j) = b(D_l) = \frac{4}{5}$ for $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq l \leq k$. In particular, each G_i has exactly one 1-triangle, and (iii) holds.

Since $b(H_j) = \frac{4}{5}$ for $1 \leq j \leq n$, it follows from Theorem 1.1 that $H_j \in \{F_1, \dots, F_7\}$ for all $1 \leq j \leq n$. Suppose $H_j \in \{F_6, F_7\}$ for some $1 \leq j \leq n$. Then $H_j = H = G$, since G is cubic and all graphs obtained from G by repeatedly applying Operations I, II and III are not cubic. So Theorem 3.3 holds, with $G \in \{F_6, F_7\}$. Therefore, we may assume that $H_j \notin \{F_6, F_7\}$ for all $1 \leq j \leq n$.

We now show that every 2-vertex of each H_j must belong to $V_1(B')$. Let u be a 2-vertex of some H_j. If u is also a 2-vertex of H, then by $(*)$, $u \in V_1(B')$. Hence, we may assume that u is a 3-vertex of H. Therefore, u is the 1-vertex in some D_l (in which case $u \in V_1(B')$) or u is contained in some fragment G_i of H. So we may assume the latter. Since G_i must contain a 1-triangle, there is a path $x_1y_1x_2y_2 \dots x_sy_su$ such that for $1 \leq p \leq s$, x_py_p belongs to a triangle $T_p := x_py_pz_px_p$, and T_1 is a 1-triangle, with $d_{G_i}(z_1) = 2$ say. By exactly the same argument as the above proof for $k_i \in \{0, 1\}$, we can show that $y_p \in V_2(B')$ for all $1 \leq p \leq s$. Therefore, since $y_su \in E(B')$ and $V_1(B')$ is independent, $u \in V_1(B')$.

Thus, since $V_1(B')$ is independent in H, no two 2-vertices of any H_j can be adjacent. So $H_j \in \{F_3, F_4, F_5\}$, and (iv) holds. ∎

It is easy to see that there are infinitely many irreducible graphs satisfying properties (i)–(iv) in Theorem 3.3. So we deduce from Theorem 3.3 that there exist infinitely many cubic graphs in $\mathcal{H}$. It remains the same even if we insist that each triangle be contained in some K_4^-. For any such graph, we only need to apply Operations I and II; and it follows from Theorem 3.3 that every fragment of the resulting irreducible graph consists of exactly three blocks: one triangle and two edges.

The next result says that as long as a subcubic graph satisfies (ii), (iii) and (iv) of Theorem 3.3, it also admits a biased maximum bipartite subgraph B satisfying (i) of Theorem 3.3.

Theorem 3.4 *Let H be a connected subcubic graph such that every block is a triangle, or a K_3^+, or a triangle-free block. Suppose H contains no 2-triangles, every fragment of H is simple and has exactly one 1-triangle, and every 2-connected triangle-free block of H belongs to $\{F_3, F_4, F_5\}$. Then $b(H) = \frac{4}{5}$, and H contains a maximum bipartite subgraph B such that $V_1(B)$ is independent and every vertex in $V_2(B)$ is a 3-vertex in H. Moreover, B can be found in $O(|V(H)|)$ time.*

Proof. Let $G_1, \dots, G_m$ (if any) denote the fragments of H, let $H_1, \dots, H_n$ (if any) denote the 2-connected triangle-free blocks of H, and let $D_1, \dots, D_k$ (if any) denote the connected subgraphs of H each of which is induced by edges incident with vertices of some K_3^+.

Since each G_i is simple and contains exactly one 1-triangle, $b(G_i) = \frac{4}{5}$ by Lemma 3.2. Since each $H_j \in \{F_3, F_4, F_5\}$, $b(H_j) = \frac{4}{5}$ (Theorem 1.1). Then $b(H) = \frac{4}{5}$ follows immediately, because $b(D_l) = \frac{4}{5}$ for $1 \leq l \leq k$.

We now prove the existence of B. First, note that each $H_j \in \{F_i : 3 \leq i \leq 5\}$, and hence admits a maximum bipartite subgraph B_j such that $V_1(B_j)$ is independent in H_j and every vertex in $V_2(B_j)$ is a 3-vertex in H_j. See Figure 1.

Secondly, it is clear that each D_l admits a maximum bipartite subgraph B_l' such that $V_1(B_l')$ is independent in D_l and every vertex in $V_2(B_l')$ is a 3-vertex in D_l.

Next, we show that each G_i admits a maximum bipartite subgraph B_i^* such that $V_1(B_i^*)$ is independent in G_i and every vertex in $V_2(B_i^*)$ is a 3-vertex in G_i. Clearly, all cut edges of G_i

must be contained in every maximum bipartite subgraph of G_i. Let B_i^* denote the breadth-first tree of G_i starting from the unique 2-vertex in G_i. Since each block of G_i is a triangle or a K_2, B_i^* is uniquely defined. Note that B_i^* is a bipartite subgraph of G_i, B_i^* uses all cut edges of G_i, and each triangle of G_i has exactly two edges in B_i^*. So B_i^* is a maximum bipartite subgraph of G_i. Without lost of generality, we may assume $V_1(B_i^*)$ contains the 2-vertex in the unique 1-triangle in G_i. Then clearly, $V_1(B_i^*)$ is independent in G_i.

It is now easy to see that B, the union of all B_j, B_l', B_i^*, gives the desired bipartite subgraph of H. Note that the bipartite graphs B_j and B_l' can be found in constant time. Since a breath-first tree can be found in linear time, the bipartite graphs B_i^* can be found on $O(|V(G_i)|)$ time. So B can be found in $O(|V(H)|)$ time. ∎

We point out here that given a connected subcubic graph H it takes $O(|V(H)|)$ time to check whether it satisfies (ii), (iii) and (iv) of Theorem 3.3. This observation, together with the above theorem, will be used in the next section to decide whether a cubic graph belongs to $\mathcal{H}$.

Corollary 3.5 *Every cubic graph in $\mathcal{H}$ can be obtained from an irreducible one satisfying (ii), (iii) and (iv) in Theorem 3.3 by applying a sequence of inverse operations of Operations I, II, and III.*

4 Membership in $\mathcal{H}$

In this section, we prove Theorem 1.3. First, we give a more detailed study on the effect that Operation III has on the bipartite density.

Lemma 4.1 *Let G be a connected subcubic graph, T a 0-triangle in G, and $N(T) = \{u, v, w\}$. Let $G' := (G - V(T)) + uv$. Suppose there exists $B \in \mathcal{B}(G)$ such that either $\{u, v, w\} \subseteq V_i(B)$ for some $i \in \{1, 2\}$, or $\{u, v\} \not\subseteq V_i(B)$ for any $i \in \{1, 2\}$. Then*

(i) $b(G) > \frac{4}{5}$ implies $b(G') > \frac{4}{5}$, and

(ii) $b(G) = \frac{4}{5}$ implies $b(G') = \frac{4}{5}$.

Proof. Let $V(T) = \{x, y, z\}$ such that $ux, vy, zw \in E(G)$, and let $A := A(x, y, z)$.

We claim that $b(G') \geq (\varepsilon(B) - 4)/(\varepsilon(G) - 5)$. Suppose $\{u, v, w\} \subseteq V_i(B)$ for some $i \in \{1, 2\}$. Then $(B - V(T)) + (A - \{xy, zw\})$ is a bipartite subgraph of G. Hence, since $B \in \mathcal{B}(G)$, B must use at least four edges from A. This in turn implies that at least one vertex in $V(T)$ belongs to $V_j(B)$, for each $j \in \{1, 2\}$ (since we assume $\{u, v, w\} \subseteq V_i(B)$). Therefore, B uses exactly four edges from A, and $B' := B - V(T)$ is a bipartite subgraph of G' such that $\varepsilon(B') = \varepsilon(B) - 4$. Hence, $b(G') \geq \varepsilon(B')/\varepsilon(G') \geq (\varepsilon(B) - 4)/(\varepsilon(G) - 5)$. Now assume $\{u, v\} \not\subseteq V_i(B)$ for any $i \in \{1, 2\}$. We may assume $u \in V_1(B)$ and $v \in V_2(B)$, and we may further assume by symmetry that $w \in V_1(B)$. Then $(B - V(T)) + (A - \{xz\})$ is a bipartite subgraph of G. Hence, B uses exactly 5 edges from A. So $B' := (B - V(T)) + uv$ is a bipartite subgraph of G', and $\varepsilon(B') = \varepsilon(B) - 4$. Again, $b(G') \geq \varepsilon(B')/\varepsilon(G') \geq (\varepsilon(B) - 4)/(\varepsilon(G) - 5)$.

If $b(G) > \frac{4}{5}$, then $\varepsilon(B) > \frac{4}{5}\varepsilon(G)$, and hence, $b(G') > \frac{4}{5}$ and (i) holds.

Now suppose $b(G) = \frac{4}{5}$. Then $\varepsilon(B) = \frac{4}{5}\varepsilon(G)$, and so, $b(G') \geq \frac{4}{5}$. Note that $\varepsilon(G)$ (and hence $\varepsilon(G')$) is divisible by 5. Suppose $b(G') > \frac{4}{5}$. Let $B' \in \mathcal{B}(G')$. Then $\varepsilon(B') > \frac{4}{5}\varepsilon(G')$. Since

$\varepsilon(G')$ is divisible by 5, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') + 1$. Without loss of generality, assume $u \in V_1(B')$. Then $v \in V_2(B')$ if $uv \in E(B')$, and $v \in V_1(B')$ if $uv \notin E(B')$ (by the maximality of B'). Define

$$B^* := \begin{cases} (B' - uv) + (A - \{xz\}), & \text{if } uv \in E(B') \text{ and } w \in V_1(B'); \\ (B' - uv) + (A - \{yz\}), & \text{if } uv \in E(B') \text{ and } w \in V_2(B'); \\ B' + (A - \{xy, wz\}), & \text{if } uv \notin E(B') \text{ and } w \in V_1(B'); \\ B' + (A - \{xy\}), & \text{if } uv \notin E(B') \text{ and } w \in V_2(B'). \end{cases}$$

Then B^* is a bipartite subgraph of G, and $\varepsilon(B^*) \geq \varepsilon(B') + 4 \geq \frac{4}{5}\varepsilon(G') + 5 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. Hence $b(G') = \frac{4}{5}$, and (ii) holds. ∎

Lemma 4.2 *Let G be a connected subcubic graph, T a 0-triangle in G, and $N(T) = \{u, v, w\}$. Let $G' := (G - V(T)) + uv$. Then*

(i) $b(G) \leq \frac{4}{5}$ implies $b(G') \leq \frac{4}{5}$, and

(ii) $b(G) < \frac{4}{5}$ implies $b(G') < \frac{4}{5}$.

Proof. Let $V(T) = \{x, y, z\}$ such that $ux, vy, wz \in E(G)$. Let $B' \in \mathcal{B}(G')$, and assume without loss of generality that $u \in V_1(B')$. As in the proof of the previous lemma, define

$$B := \begin{cases} (B' - uv) + (A - \{xz\}), & \text{if } uv \in E(B') \text{ and } w \in V_1(B'); \\ (B' - uv) + (A - \{yz\}), & \text{if } uv \in E(B') \text{ and } w \in V_2(B'); \\ B' + (A - \{xy, wz\}), & \text{if } uv \notin E(B') \text{ and } w \in V_1(B'); \\ B' + (A - \{xy\}), & \text{if } uv \notin E(B') \text{ and } w \in V_2(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 4$.

Suppose $b(G') > \frac{4}{5}$. Then $\varepsilon(B') > \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 5)$. Hence, $\varepsilon(B) > \frac{4}{5}(\varepsilon(G) - 5) + 4 = \frac{4}{5}\varepsilon(G)$, and $b(G) > \frac{4}{5}$. This proves (i).

Now assume $b(G') \geq \frac{4}{5}$. Then $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 5)$. Hence, $\varepsilon(B) \geq \frac{4}{5}(\varepsilon(G) - 5) + 4 = \frac{4}{5}\varepsilon(G)$, and $b(G) \geq \frac{4}{5}$. This proves (ii). ∎

The example in Figure 5 (in section 2) shows that the converse of Lemma 4.2 need not be true. This is the reason that in our algorithm we (sometimes) need to apply Operation III to a 0-triangle at least two times, and (as an undesired consequence) the parameter $t(G)$ need not decrease. (This would make our algorithm run exponentially in $|V(G)|$.) Fortunately, the next lemma allows us to make sure that, when we apply Operation III to a given 0-triangle several times, at most one time $t(G)$ does not decrease.

Lemma 4.3 *Let G be a connected subcubic graph, T a 0-triangle in G, and $N(T) = \{u, v, w\}$. Let $G' := (G - V(T)) + uv$. Suppose $N(v) \cap N(w) \neq \emptyset \neq N(w) \cap N(u)$. Then*

(i) $b(G) > \frac{4}{5}$ implies $b(G') > \frac{4}{5}$,

(ii) $b(G) = \frac{4}{5}$ implies $b(G') = \frac{4}{5}$, and

(iii) $G \in \mathcal{H}$ implies that $G' \in \mathcal{H}$.

Proof. Let $V(T) = \{x, y, z\}$ such that $ux, vy, wz \in E(G)$. Let $s \in N(v) \cap N(w)$ and $t \in N(u) \cap N(w)$, possibly $s = t$. Let $A := A(x, y, z)$.

First, we prove (i) and (ii). Let $B \in \mathcal{B}(G)$. We may assume $\{u, v\} \subseteq V_1(B)$ and $w \in V_2(B)$; for otherwise, (i) (respectively, (ii)) follows from Lemma 4.1(i) (respectively, Lemma 4.1(ii)). Then $(B - V(T)) + (A - \{xy\})$ is a bipartite subgraph of G, and so B must use 5 edges from A. This shows that $z \in V_1(B)$ and $\{x, y\} \subseteq V_2(B)$. Since $w \in V_2(B)$ and by the maximality of B, at least one of s, t is contained in $V_1(B)$. We may assume by symmetry that $s \in V_1(B)$. Let B^* denote the bipartite subgraph of G such that $V_1(B^*) = (V_1(B) - \{v\}) \cup \{y\}$, $V_2(B^*) = (V_2(B) - \{y\}) \cup \{v\}$, and $E(B^*)$ consists of all edges of G with an end in $V_i(B^*)$ for each $i \in \{1, 2\}$. Then $\varepsilon(B^*) \geq \varepsilon(B)$, and hence $\varepsilon(B^*) = \varepsilon(B)$ (by maximality of B), which implies $B^* \in \mathcal{B}(G)$. However, $\{u, v\} \not\subseteq V_i(B^*)$ for any $i \in \{1, 2\}$. Hence, (i) and (ii) follows from Lemma 4.1(i) and Lemma 4.1(ii), respectively.

Now assume $G \in \mathcal{H}$. Let $B \in \mathcal{B}(G)$ such that $V_1(B)$ is independent. Note by (ii), $b(G') = \frac{4}{5}$. By Lemma 2.3, $|\{u, v, w\} \cap V_1(B)| = 2$, and if $\{u, v\} \not\subseteq V_1(B)$ then $G' \in \mathcal{H}$. So we may assume that $\{u, v\} \subseteq V_1(B)$ and $w \in V_2(B)$. Since $V_1(B)$ is independent, $s, t \in V_2(B)$. Hence, $w \in V_2(B)$, and w has more neighbors in $V_2(B)$ than in $V_1(B)$, contradicting the maximality of B. ∎

We now give an algorithm that decides whether a given cubic graph belongs to $\mathcal{H}$. The rooted tree in the algorithm records both the operations performed and the graphs resulted from these operations.

Algorithm CUBIC
Input: Cubic graph G; Rooted tree $\mathcal{T}$ with one vertex v_G, the root of $\mathcal{T}$.
Output: Yes, if $G \in \mathcal{H}$; No, if $G \notin \mathcal{H}$.

1. Search for K_4^-, 1-triangles not contained in K_4^-, and 0-triangles not contained in K_4^-. If every triangle in G induces a block of G, go to step 5.

2. If there is no K_4^- contained in G, go to step 3. If there is a subgraph S of G isomorphic to K_4^-, apply Operation I to get the graph $H := G/S$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by adding vertex v_H and edge $v_G v_H$, and label $v_G v_H$ with (I). Let $G := H$ and $\mathcal{T} := \mathcal{T}'$, and go to step 1.

3. If there is no 1-triangle in G or every 1-triangle in G is a block of G, got to step 4. If there is a 1-triangle T in G that is not a block of G, apply Operation II to get the graph $H := G - V(T)$. Let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by adding vertex v_H and edge $v_G v_H$, and label $v_G v_H$ with (II). Let $G := H$ and $\mathcal{T} := \mathcal{T}'$, and go to step 1.

4. If there is no 0-triangle in G or every 0-triangle in G induces a block of G, then go to step 5. Otherwise, choose a 0-triangle T in G that does not induce a block of G. Let $N(T) = \{u, v, w\}$.

   - If there exist two pairs of vertices in $N(T)$, say $\{v, w\}$ and $\{w, u\}$, such that $N(v) \cap N(w) \neq \emptyset \neq N(w) \cap N(u)$, then apply Operation III to G to obtain $H := (G - V(T)) + uv$. Let $\mathcal{T}'$ denote the tree obtained from $\mathcal{T}$ by adding vertex v_H and edge $v_G v_H$, and label $v_G v_H$ with (IIIa). Let $G := H$ and $\mathcal{T} := \mathcal{T}'$, and go to step 1.
   - If an edge from $N(T)$ to T, say wz, is a cut edge of G, then apply Operation III to G two times to obtain $H_1 := (H - V(T)) + uw$ and $H_2 = (H - V(T)) + vw$. Let $\mathcal{T}'$ denote the tree obtained from $\mathcal{T}$ by adding vertices v_{H_i} ($i = 1, 2$) and edges $v_G v_{H_i}$,

and label $v_G v_{H_i}$ $(i = 1, 2)$ with (IIIb). Let $\mathcal{T} := \mathcal{T}'$. For each $i \in \{1, 2\}$, let $G := H_i$ and go to step 1.

- Otherwise, apply Operation III to G three times to obtain $H_1 := (H - V(T)) + uv$, $H_2 := (H - V(T)) + vw$ and $H_3 = (H - V(T)) + wu$. Let $\mathcal{T}'$ denote the tree obtained from $\mathcal{T}$ by adding vertices v_{H_i} $(i = 1, 2, 3)$ and edges $v_G v_{H_i}$, and label $v_G v_{H_i}$ $(i = 1, 2, 3)$ with (IIIc). Let $\mathcal{T} := \mathcal{T}'$. For each $i \in \{1, 2, 3\}$, let $G := H_i$ and go to step 1.

5. All graphs are connected subcubic graphs, all blocks are either a triangle, or a K_3^+, or a triangle-free graph.

   - If $\mathcal{T}$ has a subtree $\mathcal{T}^*$ such that $\mathcal{T}^*$ contains the root of $\mathcal{T}$, each 2-vertex of $\mathcal{T}^*$ is a 2-vertex of $\mathcal{T}$, each vertex of $\mathcal{T}^*$ with degree 3 in $\mathcal{T}$ has one edge with label (IIIb) pointing away from the root, each vertex of $\mathcal{T}^*$ with degree 4 in $\mathcal{T}$ has two edges with label (IIIc) pointing away from the root, and all leaves of $\mathcal{T}^*$ are irreducible graphs belonging to $\mathcal{H}$, then output Yes.
   - Otherwise, output No.

Next we prove the correctness of ALGORITHM CUBIC.

Theorem 4.4 *Let G be a connected cubic graph. Then $G \in \mathcal{H}$ iff* ALGORITHM CUBIC *produces the tree $\mathcal{T}^*$ as described in step 5.*

Proof. Suppose $G \in \mathcal{H}$. We follow ALGORITHM CUBIC and show the existence of the tree $\mathcal{T}^*$ as described in step 5. If $\mathcal{T}$ has no vertex of degree 3 or 4, $\mathcal{T}^* = \mathcal{T}$ is the desired tree. So we may assume that $\mathcal{T}$ has a vertex of degree 3 or 4. To prove the existence of $\mathcal{T}^*$, it suffices to show that for any vertex v of $\mathcal{T}$ with degree 3 (respectively, 4) which corresponds to a graph in $\mathcal{H}$, v has one child (respectively, two children) whose corresponding graphs are in $\mathcal{H}$. Let K denote the graph in $\mathcal{H}$ corresponding to v. First, suppose v is a 3-vertex in $\mathcal{T}$. Let v_1 and v_2 be the two children of v in $\mathcal{T}$, and let H_1 and H_2 be the graphs corresponding to v_1 and v_2, respectively. Then, by the construction of $\mathcal{T}$ in step 4 of ALGORITHM CUBIC, both vv_1 and vv_2 have label (IIIb), and H_1 and H_2 are obtained from K by applying Operation III on a 0-triangle T that is incident with a cut edge of K. It follows from Lemma 2.3 that at least one of H_1 and H_2 is in $\mathcal{H}$. Now, suppose v is a 4-vertex in $\mathcal{T}$. Let v_1, v_2, v_3 denote the children of v in $\mathcal{T}$, and let H_1, H_2, H_3 denote the graphs corresponding to v_1, v_2, v_3, respectively. Then from the construction of $\mathcal{T}$ in step 4 of ALGORITHM CUBIC, the edges vv_1, vv_2 and vv_3 have label (IIIc), and H_1, H_2, H_3 are obtained from K by applying Operation III on a 0-triangle T. It follows from Lemma 2.3 that at least two of H_1, H_2, H_3 belong to $\mathcal{H}$.

Now suppose $\mathcal{T}^*$ is a subtree of $\mathcal{T}$ as described in step 5 of ALGORITHM CUBIC. We view $\mathcal{T}^*$ as a rooted tree whose root is the root of $\mathcal{T}$. We say that a vertex of $\mathcal{T}^*$ is of level k if its distance to the root in $\mathcal{T}^*$ is k, and we use $\ell(v)$ to denote the level of v. Let $\ell := \max\{\ell(v) : v \in V(\mathcal{T}^*)\}$. We next show that each graph corresponding to a vertex in $\mathcal{T}^*$ belongs to $\mathcal{H}$ and admits a maximum bipartite subgraph B such that $V_1(B)$ is independent and every vertex of $V_2(B)$ is a 3-vertex. Let r be an arbitrary vertex of $\mathcal{T}^*$. We apply induction on $\ell - \ell(r)$.

Clearly, if r is a leaf of $\mathcal{T}^*$, then its corresponding graph is irreducible and belongs to $\mathcal{H}$ (by definition of $\mathcal{T}^*$). Moreover, by Theorem 3.3, it has a maximum bipartite subgraph B such that $V_1(B)$ is independent and every vertex in $V_2(B)$ is a 3-vertex. So we may assume that r is not a leaf of $\mathcal{T}^*$. In particular, $\ell - \ell(r) > 0$. Let s be a child of r in $\mathcal{T}^*$. Let K, H denote the graphs corresponding to r, s, respectively. Since $\ell(r) < \ell(s)$ and by induction hypothesis, $H \in \mathcal{H}$ and there exists $B' \in \mathcal{B}(H)$ such that $V_1(B')$ is independent and every vertex of $V_2(B')$ is a 3-vertex in H.

Suppose the edge rs has label (I) or (II). Then r is a 2-vertex in $\mathcal{T}$. If rs has label (I) (respectively, (II)), then H is obtained from K by applying Operation I (respectively, Operation II); as in the proof of Lemma 2.1 (respectively, Lemma 2.2), we can construct, from B', a maximum bipartite subgraph B of K such that $\varepsilon(B) = \frac{4}{5}\varepsilon(G)$, $V_1(B)$ is independent, and every vertex in $V_2(B)$ is a 3-vertex of K. So $K \in \mathcal{H}$.

Now assume that rs has label (IIIa), or (IIIb), or (IIIc). Then H is obtained from K by applying Operation III on a 0-triangle in K, say T. Let $T = xyzx$ and $ux, vy, wz \in E(K)$ such that $H = (K - V(T)) + uv$. Since $V_1(B')$ is independent in H and every vertex in $V_2(B')$ is a 3-vertex in H, $w \in V_1(B')$ and either $\{u, v\} \subseteq V_2(B')$ or $\{u, v\} \not\subseteq V_j(B')$ for any $j \in \{1, 2\}$.

Suppose $\{u, v\} \subseteq V_2(B')$. Then $B := B' + (A(x, y, z) - \{xy\})$ is a bipartite subgraph of K and $\varepsilon(B) = \varepsilon(B') + 5 > \frac{4}{5}\varepsilon(K)$. This implies that rs must have label (IIIb) or (IIIc); for otherwise, $b(K) = \frac{4}{5}$ by Lemmas 4.2(ii) and 4.3(i). If rs has label (IIIb), then r is a 3-vertex in $\mathcal{T}$, and ux or vy, say ux, is a cut edge of K (see step 4 of Algorithm CUBIC), which means that $B' + uv$ is still a bipartite subgraph of H, a contradiction to $B' \in \mathcal{B}(H)$. Therefore, rs must have label (IIIc), and hence r is a 4-vertex in $\mathcal{T}^*$. Let H' denote the graph corresponding to the other child of r in $\mathcal{T}^*$. By induction, $H' \in \mathcal{H}$. Without loss of generality, we may assume that $H' := (K - V(T)) + vw$. Then $B^* := (B - V(T)) + vw$ is a bipartite subgraph of H' and $\varepsilon(B^*) = \varepsilon(B) - 4 > \varepsilon(B') = \frac{4}{5}\varepsilon(H) = \frac{4}{5}\varepsilon(H')$. This shows $b(H') > \frac{4}{5}$, a contradiction.

Now assume $\{u, v\} \not\subseteq V_j(B')$ for any $j \in \{1, 2\}$. By symmetry, we may assume $u \in V_1(B')$ and $v \in V_2(B')$. Then $B := (B' - uv) + (A(x, y, z) - \{xz\})$ is a bipartite subgraph of K such that $\varepsilon(B) = \frac{4}{5}\varepsilon(K)$, $V_1(B)$ is independent in K, and every vertex of $V_2(B)$ is a 3-vertex of K. To prove $K \in \mathcal{H}$, we need to show that $B \in \mathcal{B}(K)$. Suppose $B \notin \mathcal{B}(K)$. Then there is a bipartite subgraph B^* of K such that $\varepsilon(B^*) > \frac{4}{5}\varepsilon(K)$. Since $\varepsilon(B') = \frac{4}{5}\varepsilon(H)$, $\varepsilon(H)$ (and hence, $\varepsilon(K)$) is divisible by 5. So $\varepsilon(B^*) \geq \frac{4}{5}\varepsilon(K) + 1$. We claim that B^* uses 5 edges from $A(x, y, z)$; otherwise, $B^* \cap H$ is a bipartite subgraph of H with at least $\varepsilon(B^*) - 4 \geq \frac{4}{5}\varepsilon(K) - 3 > \frac{4}{5}\varepsilon(H)$ edges, a contradiction. Moreover, $\{u, v\} \subseteq V_j(B^*)$ for some $j \in \{1, 2\}$; for otherwise, $(B^* \cap H) + uv$ is a bipartite subgraph of H with at least $\varepsilon(B^*) - 4 \geq \frac{4}{5}\varepsilon(K) - 3 > \frac{4}{5}\varepsilon(H)$ edges, a contradiction. Therefore, $w \in V_{3-j}(B^*)$ (since B^* uses 5 edges from $A(x, y, z)$). Then an argument similar to the above paragraph gives a contradiction. ∎

We now investigate the running time of Algorithm CUBIC. Recall that for any graph G, $t(G)$ denotes the number of triangles in G that are not blocks of G and do not share edges with any other triangles in G.

Theorem 4.5 Algorithm CUBIC *runs in* $O(|V(G)|^{t(G)+1})$ *time.*

Proof. Let $f(n, t)$ be the maximum number of vertices the output tree $\mathcal{T}$ has when Algorithm CUBIC is performed on any connected cubic graph G with $|V(G)| = n$ and $t(G) = t$. Note that $f(n, t) \geq 1$.

We claim that $f(n, t) \leq f(n-3, t) + 2f(n-3, t-1)$ for $n \geq 4$ and $t \geq 1$. Let G be a connected cubic graph for which the tree $\mathcal{T}$ has $f(n, t)$ vertices. If Algorithm CUBIC executes step 2 (respectively, step 3), then the resulting graph G/S (respectively, $G - V(T)$) has $n - 3$ vertices; and hence, $f(n, t) \leq 1 + f(n - 3, t) \leq f(n - 3, t) + 2f(n - 3, t - 1)$. If Algorithm CUBIC executes step 4 and only produces one graph H, then again, H has $n-3$ vertices, and we have $f(n,t) \leq 1 + f(n-3,t) \leq f(n-3,t) + 2f(n-3,t-1)$. If Algorithm CUBIC executes step 4 and produces two graph H_1 and H_2, then $t(H_1) = t-1 = t(H_2)$, and H_1 and H_2 each have $n-3$ vertices; and we have $f(n,t) \leq 1 + 2f(n-3,t-1) \leq f(n-3,t) + 2f(n-3,t-1)$. If Algorithm CUBIC executes step 4 and produces three graphs H_1, H_2 and H_3, then each H_i has $n-3$ vertices and two of which, say H_1 and H_2, has $t(H_1) = t-1 = t(H_2)$; and hence, $f(n,t) \leq f(n-3,t) + 2f(n-3,t-1)$.

By recursively applying the inequality $f(n,t) \leq f(n-3,t) + 2f(n-3,t-1)$, we deduce $f(n,t) \leq nf(n-3,t-1)$. This implies $f(n,t) \leq n^t$.

Note that the loop of the algorithm consists of steps 1, 2, 3 and 4, and each step produces at least one new vertex for $\mathcal{T}$. Clearly, each step runs in $O(n)$ time. Moreover, by Theorem 3.4, it takes $O(n)$ time to check whether each leaf of $\mathcal{T}$ corresponds to a graph in $\mathcal{H}$ and has a bipartite subgraph B such that $V_1(B)$ is independent and every vertex in $V_2(B)$ is a 3-vertex. So Algorithm CUBIC runs in $O(|V(G)|^{t(G)+1})$ time. ∎

Clearly, if $t(G)$ is constant or not considered as part of the input, then Algorithm CUBIC runs in polynomial time. From the proof of Theorem 4.4, we can see that Algorithm CUBIC may be modified so that when $G \in \mathcal{H}$ it also produces, in $O(|V(G)|^{t(G)+1})$ time, a biased maximum bipartite subgraph of G.

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