Partitioning 3-uniform hypergraphs

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Abstract

Bollobás and Thomason conjectured that the vertices of any r-uniform hypergraph with m edges can be partitioned into r sets so that each set meets at least $rm/(2r-1)$ edges. For $r = 3$, Bollobás, Reed and Thomason proved the lower bound $(1-1/e)m/3 \approx 0.21m$, which was improved to $5m/9$ by Bollobás and Scott (while the conjectured bound is $3m/5$). In this paper, we show that this Bollobás-Thomason conjecture holds asymptotically for $r = 3$.

*Partially supported by NSA and by NSFC Project 10628102

1 Introduction

Let G be a graph or hypergraph, and let $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We write $e_G(S) := |\{e \in E(G) : e \subseteq S\}|$, $e_G(S,T) := |\{e \in E(G) : e \cap S \neq \emptyset \neq e \cap T\}|$, and $d_G(S) := |\{e \in E(G) : e \cap S \neq \emptyset\}|$. When understood, the reference to G in the subscript may be dropped.

An example of classical graph partition problems is the well known *Maximum Bipartite Subgraph Problem*: Given a graph G find a partition V_1, V_2 of $V(G)$ maximizing $e(V_1, V_2)$. There are an extensive body of work on this problem, from various perspectives [15]. Note that the Maximum Bipartite Subgraph Problem asks for a partition of an input graph that optimizes only one quantity.

Any problem that asks for partitions of graphs or hypergraphs to optimize several quantities simultaneously is said to be a *judicious* partition problem. The *Bottleneck Bipartition Problem* is one such example: Given a graph G find a partition V_1, V_2 of $V(G)$ minimizing $\max\{e(V_1), e(V_2)\}$, or equivalently, maximizing $\min\{d(V_1), d(V_2)\}$ (since $d(V_i) = |E(G)| - e(V_{3-i})$ for $i = 1, 2$). This problem was raised by Entringer, and is shown to be NP-hard in [18].

In [1] it is shown that the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem are related. Note that if V_1, V_2 is a partition of a graph G maximizing $e(V_1, V_2)$, then each $v \in V_i$ has at least as many neighbors in V_{3-i} as in V_i. So $e(V_1, V_2) \geq 2e(V_i)$ for $i = 1, 2$, which implies $e(V_i) \leq m/3$, where m is the number of edges in G. Hence $d(V_i) \geq m - m/3 = 2m/3$ for $i = 1, 2$. In an attempt to extend this to hypergraphs, Bollobás and Thomason made the following conjecture for hypergraphs; see [7].

Conjecture 1.1 (*Bollobás and Thomason*) *For any integer $r \geq 3$, the vertex set of any r-uniform hypergraph with m edges admits a partition $V_1, \ldots, V_r$ such that for $i = 1, \ldots, m$,*

$$d(V_i) \geq \frac{r}{2r-1} m.$$

The conjectured bound is the best possible for complete r-uniform graphs on $2r-1$ vertices. To see this, note that such a graph has $m = \binom{2r-1}{r}$ edges, and any r-partition of such a graph has a partition set with just one vertex, which meets $\binom{2r-2}{r-1}$ edges.

Bollobás, Reed and Thomason [3] proved that every 3-uniform hypergraph with m edges has a partition V_1, V_2, V_3 such that $d(V_i) \geq (1 - 1/e)m \approx 0.21m$ (here e is the base of the natural logrithm). In [7], this bound is improved to $5m/9$ by Bollobás and Scott. Note that the bound for $r = 3$ in Conjecture 1.1 is $3m/5$. In this paper, we prove the following result, which shows that Conjecture 1.1 holds asymptotically for $r = 3$.

Theorem 1.2 *Every 3-uniform hypergraph with m edges has a partition into sets V_1, V_2, V_3 such that for $i = 1, 2, 3$,*

$$d(V_i) \geq 3m/5 + o(m).$$

We use an approach developed by Bollobás and Scott [5, 8]. The idea is to partition the large degree vertices first, and then partition the remaining vertices using a random process. The key is to find appropriate probabilities for this random process which result in the desired bounds on the expectations $d(V_i)$. An application of Azuma-Hoeffding inequality allows us to bound the deviations from these expectations.

We organize our paper as follows. In Section 2, we state the main lemma, Lemma 2.1, and use it to prove Theorem 1.2. The proof of the main lemma is carried out in Sections 3, 4 and 5. In the main lemma, we need to bound three quantities simultaneously. In Section 3, we prove two lemmas that can be used to bound two quantities simultaneously. These lemmas will then be used in Section 4 to treat two special cases of the main lemma. We conclude the proof of the main lemma in Section 5.

2 Proof of Theorem 1.2

We first state the following lemma; its proof is the work of the rest of this paper. Let $\mathbb{R}^+$ denote the set of nonnegative reals, and let $\mathbb{N}^+$ denote the set of natural numbers.

Lemma 2.1 *Let $b_{ij}, x_i, a_i, c \in \mathbb{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. Then there exist $p_1, p_2, p_3 \in [0,1]$ with $p_1 + p_2 + p_3 = 1$ such that for any $\{i,j,k\} = \{1,2,3\}$,*

$$(1-p_i)(b_{jk} + x_j + x_k) + (1-p_i)^2(a_j + a_k) + (1-p_i)^3 c \le 2/5.$$

We also need the following lemma, which is easy to prove. Let G be a graph and let $w : E(G) \to \mathbb{R}^+$. For any $S \subseteq V(G)$, we write $w(S) = \sum_{e \subseteq S} w(e)$. For any $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we use (S,T) to denote the set of edges st with $s \in S$ and $t \in T$; and write $w(S,T) = \sum_{e \in (S,T)} w(e)$.

Lemma 2.2 *Let G be a graph and let $w : E(G) \to \mathbb{R}^+$, and let $V(G) = V_1 \cup \ldots \cup V_k$ be a k-partition minimizing $\sum_{i=1}^{k} w(V_i)$. Then for any $1 \le i \ne j \le k$*

$$w(V_i, V_j) \ge \max\{2w(V_i), 2w(V_j)\}.$$

Proof. For any $v \in V_i$ and for any $j \in \{1, \ldots, k\} \setminus \{i\}$, we have

$$\sum_{\{u \in V_i - v : uv \in E(G)\}} w(vu) \le \sum_{\{u \in V_j : uv \in E(G)\}} w(vu).$$

Summing over $v \in V_i$, we get $2w(V_i) \le w(V_i, V_j)$. ∎

Finally we need the Azuma-Heoffding inequality [2, 12], to bound deviations. We use the version given in [5].

Lemma 2.3 *Let $Z_1, \ldots, Z_n$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z := (Z_1, \ldots, Z_n)$, and let $f : \{1, \ldots, k\}^n \to \mathbb{N}$ such that $|f(Y) - f(Y')| \le c_i$ for any $Y, Y' \in \{1, \ldots, k\}^n$ which differ only in the ith coordinate. Then for any $z > 0$,*

$$\mathbb{P}\left(f(Z) \ge E(f(Z)) + z\right) \le \exp\left(\frac{-z^2}{2\sum_{i=1}^{k} c_i^2}\right),$$

$$\mathbb{P}\left(f(Z) \le E(f(Z)) - z\right) \le \exp\left(\frac{-z^2}{2\sum_{i=1}^{k} c_i^2}\right).$$

Now Theorem 1.2 follows from the following result.

Theorem 2.4 *Let G be a 3-uniform hypergraph with m edges. Then there is a 3-partition $V(G) = V_1 \cup V_2 \cup V_3$ such that, for $i = 1, 2, 3$,*

$$d(V_i) \geq \frac{3}{5}m + O(m^{\frac{6}{7}}).$$

Proof. We may assume that G is connected; as otherwise, we may simply consider the individual components. Hence every vertex of G has positive degree.

Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $U_1 := \{v_1, \ldots, v_t\}$ and $U_2 := V(G) \setminus U_1$, with $t = \lfloor m^\alpha \rfloor$ and $0 < \alpha < 1/3$. Thus $t < n$ (since $m < n^3$). Moreover,

$$m^\alpha d(v_{t+1}) \leq \sum_{v \in U_1} d(v) < \sum_{v \in V(G)} d(v) = 3m;$$

so $d(v_{t+1}) < 3m^{1-\alpha}$. Hence

$$\sum_{i=t+1}^{n} d(v_i)^2 < 3m^{1-\alpha} \sum_{i=1}^{n} d(v_i) = 9m^{2-\alpha}.$$

For any 3-partition $U_1 = X_1 \cup X_2 \cup X_3$ and for $1 \leq i \neq j \leq 3$, define

$$\begin{aligned}
x_i &= |\{e \in E(G) : |e \cap X_i| = 2, |e \cap U_2| = 1\}|, \\
a_i &= |\{e \in E(G) : |e \cap X_i| = 1, |e \cap U_2| = 2\}|, \\
b_{ij} &= |\{e \in E(G) : |e \cap X_i| = |e \cap X_j| = |e \cap U_2| = 1\}|, \\
c &= |\{e \in E(G) : |e \cap U_2| = 3\}|.
\end{aligned}$$

Then $m = e(U_1) + b_{12} + b_{23} + b_{13} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c$.

By Lemma 2.2, we may choose the 3-partition $U_1 = X_1 \cup X_2 \cup X_3$ such that for $1 \leq i \neq j \leq 3$,

$$b_{ij} \geq \max\{2x_i, 2x_j\}.$$

For $1 \leq i \leq 3$, assign color i to the vertices in X_i. We extend the coloring to U_2 as follows: each vertex in U_2 is independently colored i with probability p_i for $1 \leq i \leq 3$.

For $i = 1, 2, 3$, let V_i be the vertices with color i, and let

$$y_i = |\{e \in E(G) : e \subseteq U_1 \text{ and } e \cap X_i \neq \emptyset\}.$$

Then, for any permutation ijk of $\{1, 2, 3\}$,

$$\mathbb{E}(d(V_i)) = b_{ij} + b_{ik} + x_i + a_i + p_i(b_{jk} + x_j + x_k) + (1 - (1 - p_i)^2)(a_j + a_k) + (1 - (1 - p_i)^3)c + y_i.$$

Thus

$$f_i := m - \mathbb{E}(d(V_i)) - e(U_1) + y_i = (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c,$$

and

$$\alpha := m - e(U_1) = b_{12} + b_{23} + b_{31} + a_1 + a_2 + a_3 + x_1 + x_2 + x_3 + c.$$

By applying Lemma 2.1 (with $b_{ij}/\alpha, a_i/\alpha, x_i/\alpha, c/\alpha$ as b_{ij}, a_i, x_i, c, respectively), there exist $p_i \in [0,1]$ with $p_1+p_2+p_3=1$ such that for any $1 \le i \le 3$, $f_i/\alpha \le 2/5$. So

$$f_i \le \frac{2}{5}(m - e(U_1)).$$

Hence

$$\mathbb{E}(d(V_i)) = m - f_i - e(U_1) + y_i \ge \frac{3}{5}m - \frac{3}{5}e(U_1) + y_i.$$

Changing the color of any v_j, $t+1 \le j \le n$, affects $d(V_i)$ by at most $d(v_j)$. So by Lemma 2.3, we have for $i = 1,2,3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) \le \exp\left(\frac{-z^2}{2\sum_{j=t+1}^{n} d(v_j)^2}\right) < \exp\left(\frac{-z^2}{18m^{2-\alpha}}\right).$$

Taking $z = \sqrt{18\ln 3}\, m^{1-\alpha/2}$, we have for $i = 1,2,3$,

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) < \frac{1}{3}.$$

Therefore, there exists a partition $V(G) = V_1 \cup V_2 \cup V_3$ such that for $i = 1,2,3$,

$$d(V_i) \ge \mathbb{E}(d(V_i)) - z \ge \frac{3}{5}m - \frac{3}{5}e(U_1) + y_i - z \ge \frac{3}{5}m - \frac{3}{5}e(U_1) - z.$$

Since $|U_1| = t \le m^\alpha$, $e(U_1) = O(m^{3\alpha})$. So

$$\frac{3}{5}e(U_1) + z = O(m^{3\alpha}) + \sqrt{18\ln 2}\, m^{1-\alpha/2}.$$

Choosing $\alpha = \frac{2}{7}$ to minimize $\max\{3\alpha, 1-\alpha/2\}$, we have the desired bound. ∎

3 Bounding two quantities

In this section, we prove a lemma to be used in our proof of Lemma 2.1.

First, we prove the following lemma (to be used in the proof of Lemma 4.1), which is a slight variation of the main lemma in [5]. The difference is that here we relax the constraint $z \ge \max\{2x, 2y\}$ in [5] to $z \ge x+y$, and as a consequence we have a weaker bound. Our proof mimics that in [5], where a more general result is proved.

Lemma 3.1 *Let $a,b,x,y,z,e \in \mathbb{R}^+$ such that $z \ge x+y$ and $a+b+x+y+z+e = 1$. Then there exists $p \in (0,1)$ such that*

$$p^2a + px + p^3e \le 1/6, \text{ and } (1-p)^2b + (1-p)y + (1-p)^3e \le 1/6.$$

Proof. For convenience, let

$$f_1 := p^2a + px + p^3e, \text{ and } f_2 := (1-p)^2b + (1-p)y + (1-p)^3e.$$

Note that f_1 and f_2 are continuous functions of p on $[0,1]$. We may assume that

(1) $a + x + e > 0$ and $b + y + e > 0$.
Otherwise, by symmetry, we may assume $a + x + e = 0$. Since f_1 and f_2 are continuous functions of p, there exist $0 < \epsilon < 1$ such that

$$|f_1(\epsilon) - f_1(1)| < 1/6, \text{ and } |f_2(\epsilon) - f_2(1)| < 1/6.$$

Since $f_1(1) = a + x + e = 0$, we have $f_1(\epsilon) < 1/6$. Also since $f_2(1) = 0$, we have $f_2(\epsilon) < 1/6$. So letting $p = \epsilon$, the assertion of the lemma holds. Thus we may assume (1).

By (1), $f_1(1) = a+x+e > 0$ and $f_2(0) = b+y+e > 0$. Therefore, since $f_1(0) = 0 = f_2(1)$, and $f_1(p)$ is increasing and continuous on $[0, 1]$ and $f_2(p)$ is decreasing and continuous on $[0, 1]$, we have
(2) for any a, b, x, y, z, e satisfying (1), there exists $p \in (0, 1)$ such that $f_1 = f_2$.

We call $\mathbf{v} := (a, b, x, y, z, e, p) \in [0, 1]^7$ a *satisfying point* if $a, b, x, y, z, e, p \in \mathbb{R}^+$, $a+b+x+y+z+e = 1$, $z \geq x + y$, $p \in [0, 1]$, and $f_1 = f_2$. Let $\mathscr{D}$ denote the set of all satisfying points. Note that $\mathscr{D}$ is a compact subset of $[0, 1]^7$. A point in $\mathscr{D}$ is said to be a *maximal point* if the value of f_1 at that point is the maximum of f_1 over $\mathscr{D}$. Let $\mathscr{M}$ be the set of maximal points, which is nonempty since $\mathscr{D} \neq \emptyset$ (by (1) and (2)) and $\mathscr{D}$ is compact.

It then suffices to show that $f_1(\mathbf{v}) \leq 1/6$ for all $\mathbf{v} \in \mathscr{M}$. We do so by looking for a special maximal point.

(3) There exists $(a, b, x, y, z, e, p) \in \mathscr{M}$ such that $e = 0$, $z = x + y$, and $ab = 0$.
Let $\mathbf{v} := (a, b, x, y, z, e, p) \in \mathscr{M}$. If $e > 0$, then let $\mathbf{v}' := (a + pe, b + (1 - p)e, x, y, z, 0, p)$. It is easy to check that $\mathbf{v}' \in \mathscr{D}$ and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$. Hence $\mathbf{v}' \in \mathscr{M}$, since $\mathbf{v} \in \mathscr{M}$ and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. So we may assume $e = 0$.

Let $u := x + y$. Suppose $z > u$. Let $\mathbf{v}' := (a + z - u, b, x, y, u, 0, p')$ with $p' \in [0, 1]$, which satisfies (1). So by (2), we may choose $p' \in (0, 1)$ so that $f_1(\mathbf{v}') = f_2(\mathbf{v}')$; then $\mathbf{v}' \in \mathscr{D}$. If $p' < p$, then $f_2(\mathbf{v}') > f_2(\mathbf{v})$, contradicting the assumption that $\mathbf{v} \in \mathscr{M}$. So $p' \geq p$. Then

$$\begin{aligned} f_1(\mathbf{v}') - f_1(\mathbf{v}) &\geq p^2(z - u) > 0, \quad \text{and} \\ f_2(\mathbf{v}') - f_2(\mathbf{v}) &\leq b((1 - p')^2 - (1 - p)^2) + y((1 - p') - (1 - p)) = -(p' - p)((2 - p - p')b + y) \leq 0. \end{aligned}$$

Hence $f_1(\mathbf{v}') > f_1(\mathbf{v}) = f_2(\mathbf{v}) \geq f_2(\mathbf{v}')$, a contradiction.

Therefore, we may assume $z = x + y$. Suppose $a > 0$ and $b > 0$. Let $\varepsilon = \min\{pa, (1 - p)b\}$, and let

$$\mathbf{v}' = (a', b', x', y', z', e', p') := (a - \frac{\varepsilon}{p}, b - \frac{\varepsilon}{1 - p}, x + \varepsilon, y + \varepsilon, z + 2\varepsilon, 0, p).$$

It is easy to see that $e' = 0$, $z' = x' + y'$, and $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2$ (and hence $f_1(\mathbf{v}') = f_2(\mathbf{v}')$). Since $a + b + x + y + z = 1$,

$$a' + b' + x' + y' + z' = 1 + 4\varepsilon - \left(\frac{\varepsilon}{p} + \frac{\varepsilon}{1 - p}\right).$$

By Cauchy-Schwarz,

$$4\varepsilon \leq \frac{\varepsilon}{p} + \frac{\varepsilon}{1 - p}$$

So we have $a' + b' + x' + y' + z' \leq 1$.

If $a' + b' + x' + y' + z' = 1$ then $\mathbf{v}' \in \mathscr{D}$. Since $f_i(\mathbf{v}') = f_i(\mathbf{v})$, we have $\mathbf{v}' \in \mathscr{M}$, and (3) holds with $\mathbf{v}'$. We may thus assume that $a' + b' + x' + y' + z' < 1$. Let

$$\alpha = \frac{\varepsilon}{p} + \frac{\varepsilon}{1-p} - 4\varepsilon,$$

and let

$$\mathbf{v}'' := (a'', b'', x'', y'', z'', e'', p'') = (a' + \alpha, b', x', y', z', 0, p'')$$

for some $p'' \in (0,1)$. Note that $e'' = 0$, $z'' = x'' + y''$, $a'' + b'' + x'' + y'' + z'' = 1$, and $\mathbf{v}''$ satisfies (1). So by (2), we may choose $p'' \in (0,1)$ such that $f_1(\mathbf{v}'') = f_2(\mathbf{v}'')$, and hence $\mathbf{v}'' \in \mathscr{D}$.

If $p'' \geq p'$ then $f_1(\mathbf{v}'') > f_1(\mathbf{v}') = f_1(\mathbf{v})$ since $a'' > a'$. If $p'' < p'$ then $f_2(\mathbf{v}'') > f_2(\mathbf{v}') = f_2(\mathbf{v})$. In either case, we obtain a contradiction to the assumption that $\mathbf{v} \in \mathscr{M}$.

Let $\mathscr{M}' = \{(a, b, x, y, z, e, p) \in \mathscr{M} : a = b = e = 0 \text{ and } z = x + y\}$. We may assume that

(4) $\mathscr{M}' = \emptyset$.

For otherwise, let $\mathbf{v} = (0, 0, x, y, x + y, 0, p) \in \mathscr{M}'$. Then $f_1(\mathbf{v}) = px$ and $f_2(\mathbf{v}) = (1 - p)y$. Moreover, $x + y = 1/2$. Since $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we have $px = (1-p)(1/2 - x)$. Hence, $p = 1 - 2x$, and $f_1(\mathbf{v}) = x(1 - 2x) = 1/8 - 2(1/4 - x)^2 \leq 1/8 < 1/6$. So the assertion of the lemma holds; and thus we may assume (4).

By (3), let $\mathbf{v} = (a, b, x, y, z, e, p) \in \mathscr{M}$ such that $e = 0$, $z = x + y$, and $a = 0$ or $b = 0$. By (4), we have $a \neq 0$ or $b \neq 0$. So by symmetry, we may assume $a = 0$ and $b \neq 0$. Then $b + 2(x + y) = 1$, and hence $x = (1 - b)/2 - y$. So

$$f_1(\mathbf{v}) = xp = \frac{(1-b)p}{2} - yp, \text{ and } f_2(\mathbf{v}) = y(1-p) + b(1-p)^2.$$

Since $\mathbf{v} \in \mathscr{M}$, $f_1(\mathbf{v})$ is the maximum value of f_1 over $\mathscr{D}$ subject to $g := f_1 - f_2 = 0$, where f_1, f_2, g are considered as functions of b, y, p.

Case 1. $y \neq 0$.

Then $y \in (0,1)$ and $b \in (0,1)$; so $\mathbf{v}$ is a critical point of f_1 (as a function of b, y). Hence $\mathbf{v}$ must satisfy $\partial f_1/\partial b = \lambda \partial g/\partial b$ and $\partial f_1/\partial y = \lambda \partial g/\partial y$, where λ is a Lagrange multiplier. So we have

$$p = \lambda \left(p + 2(1-p)^2\right), \text{ and } p = \lambda \left(p + (1-p)\right) = \lambda.$$

Since $p \in (0,1)$, we have $\lambda \neq 0$. So from the above equations we deduce that $(1 - p) = 2(1-p)^2$. Again since $p \neq 1$, we have $p = 1/2$. Let

$$\mathbf{v}' := (a', b', x', y', z', e', p') = (0, 0, x, y + b/2, z + b/2, 0, p).$$

Then $a' + b' + x' + y' + z' + e' = 1$, $z' = x' + y'$, and $f_1(\mathbf{v}') = f_1(\mathbf{v})$. Since $p = 1/2$,

$$f_2(\mathbf{v}') = (1-p)(y + b/2) = (1-p)y + (1-p)b/2 = (1-p)y + (1-p)^2 b = f_2(\mathbf{v}).$$

This implies $\mathbf{v}' \in \mathscr{M}'$, contradicting (4).

Case 2. $y = 0$.

Then $f_1(\mathbf{v}) = (1-b)p/2$ and $f_2(\mathbf{v}) = b(1-p)^2$. Note that $b \in (0,1)$ and $p \in (0,1)$. Since $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$ subject to $g := f_1 - f_2 = 0$ (considered as functions of p and b), $\mathbf{v}$ satisfies $\partial f_1/\partial p = \lambda \partial g/\partial p$ and $\partial f_1/\partial b = \lambda \partial g/\partial b$. Therefore,

$$(1-b)/2 = \lambda\left((1-b)/2 + 2b(1-p)\right), \text{ and } p/2 = \lambda\left(p/2 + (1-p)^2\right).$$

Since $p \in (0,1)$, we have $\lambda \neq 0$; so we derive from the above equations that $b = (1-p)/(1+p)$. From $f_1(\mathbf{v}) = f_2(\mathbf{v})$, we deduce $b = \frac{p}{p+2(1-p)^2}$. Hence we have

$$\frac{p}{p+2(1-p)^2} = \frac{1-p}{1+p}.$$

Simplifying this we get $p^3 - 2p^2 + 3p - 1 = 0$, which implies $p < 1/2$; since the function $p^3 - 2p^2 + 3p - 1$ is increasing when $p \geq 1/2$ and takes value $1/8$ when $p = 1/2$.

We now claim that $f_1 \leq 1/6$. For otherwise, we have $f_1 > 1/6$, i.e.,

$$\frac{(1-b)p}{2} = \frac{p^2}{(1+p)} > 1/6.$$

But this gives $p > 1/2$, a contradiction. ∎

In the next lemma we show that one may choose p so that two quantities are equal and (as in the previous lemma) bounded from above. The proof is similar to that of Lemma 3.1.

Lemma 3.2 *Let $\mathscr{D}$ denote the set of all points (a,b,x,y,e,p) such that $a,b,x,y,e \in \mathbb{R}^+$, $p \in (1/5,1)$, $a+b+2(x+y+e) = 1$, and $p^2a + px + p^3e = (6/5-p)^2b + (6/5-p)y + (6/5-p)^3e$. Suppose $\mathscr{D} \neq \emptyset$. Then for any $(a,b,x,y,e,p) \in \mathscr{D}$, $p^2a + px + p^3e \leq 9/50$.*

Proof. For convenience, let

$$\begin{aligned} g_1(a,b,x,y,e,p) &:= p^2a + px + p^3e, \text{ and} \\ g_2(a,b,x,y,e,p) &:= (6/5-p)^2b + (6/5-p)y + (6/5-p)^3e. \end{aligned}$$

A point $\mathbf{v} := (a,b,x,y,e,p) \in \mathscr{D}$ is said to be *maximal* if $g_1(\mathbf{v})$ is the maximum of g_1 over $\mathscr{D}$. Let $\mathscr{M}$ denote the set of all maximal points. Since $\mathscr{D}$ is compact and $\mathscr{D} \neq \emptyset$, $\mathscr{M} \neq \emptyset$. We claim that

(1) for any $\mathbf{v} = (a,b,x,y,e,p) \in \mathscr{M}$, $e = 0$.
For, suppose $\mathbf{v} := (a,b,x,y,e,p) \in \mathscr{M}$ and $e \neq 0$. Let

$$\mathbf{v}' := (a',b',x',y',e',p') = (a+pe, b+(6/5-p)e, x, y, 0, p).$$

Then

$$g_1(\mathbf{v}') = p^2a' + px = g_1(\mathbf{v}), \text{ and } g_2(\mathbf{v}') = (6/5-p)^2b' + (6/5-p)y = g_2(\mathbf{v}).$$

Note that $a' + b' + 2(x' + y' + e') = a + b + 2(x+y) + 6e/5 = 1 - 4e/5$. Let

$$e_1 = \frac{(6/5-p)^2}{p^2 + (6/5-p)^2} \cdot \frac{4e}{5}, \text{ and } e_2 = \frac{p^2}{p^2 + (6/5-p)^2} \cdot \frac{4e}{5}.$$

Then $e_1 + e_2 = 4e/5$ and $p^2 e_1 = (6/5 - p)^2 e_2$. Let

$$\mathbf{v}'' := (a'', b'', x'', y'', e'', p'') = (a' + e_1, b' + e_2, x', y', e', p').$$

Then $a'' + b'' + 2(x'' + y'' + e'') = a' + b' + 2(x' + y' + e') + e_1 + e_2 = 1$. Moreover,

$$g_1(\mathbf{v}'') = g_1(\mathbf{v}') + p^2 e_1 = g_2(\mathbf{v}') + (6/5 - p)^2 e_2 = g_2(\mathbf{v}'').$$

So $\mathbf{v}'' \in \mathscr{D}$. This is a contradiction to the facts that $g_1(\mathbf{v}'') > g_1(\mathbf{v}') = g_1(\mathbf{v})$ and $\mathbf{v} \in \mathscr{M}$, completing the proof of (1).

Let $\mathscr{M}' = \{(a, b, x, y, e, p) \in \mathscr{M} : x = y = e = 0\}$. We may assume
(2) $\mathscr{M}' = \emptyset$.
For, suppose there exists some $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}'$. Then $a + b = 1$,

$$g_1(\mathbf{v}) = p^2 a, \text{ and } g_2(\mathbf{v}) = (6/5 - p)^2 b.$$

Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p^2}{p^2 + (6/5 - p)^2}.$$

Note that for any $s, t \in \mathbb{R}^+$, we have $2\sqrt{st} \le s + t$ and $2st \le s^2 + t^2$; so $8s^2t^2 \le (s+t)^2(s^2+t^2)$. So for $s, t > 0$, we have

$$\frac{s^2 t^2}{s^2 + t^2} \le \frac{1}{2}\left(\frac{s+t}{2}\right)^2.$$

Thus

$$g_1(\mathbf{v}) = g_2(\mathbf{v}) = \frac{p^2\,(6/5 - p)^2}{p^2 + (6/5 - p)^2} \le \frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{9}{50},$$

and the assertion of the lemma holds. So we may assume (2).

By (1) and (2), there exists $\mathbf{v} = (a, b, x, y, e, p) \in \mathscr{M}$ such that $e = 0$, and $x \neq 0$ or $y \neq 0$. We now show that $\mathbf{v}$ may be chosen so that
(3) $y = 0$.
For, suppose $y \neq 0$. Since $a + b + 2(x + y + e) = 1$ and $e = 0$, $x = (1 - a - b - 2y)/2$. So

$$g_1(\mathbf{v}) = p^2 a + p\frac{1 - a - b - 2y}{2}, \text{ and}$$
$$g_2(\mathbf{v}) = (6/5 - p)^2\, b + (6/5 - p)\, y.$$

Suppose $b \neq 0$. Then since we assume $y \neq 0$ and because $\mathbf{v} \in \mathscr{M}$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of b and y. By applying the method of Lagrange multipliers, we have $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial y = \lambda \partial g/\partial y$. Hence

$$-\frac{p}{2} = \lambda\left(-\frac{p}{2} - (6/5 - p)^2\right), \text{ and } -p = \lambda\left(-p - (6/5 - p)\right).$$

Since $p \in (1/5, 1)$, $\lambda \neq 0$. Hence from the above expressions we deduce that $(6/5 - p)^2 = (6/5 - p)/2$. So $p = \frac{7}{10}$, since $p \in (1/5, 1)$. Let

$$\mathbf{v}' := (a', b', x', y', e', p') = (a, b + 2y, x, 0, 0, p).$$

Then

$$\begin{aligned}
&a' + b' + 2(x' + y' + e') = a + b + 2(x + y) = 1,\\
&g_1(\mathbf{v}') = p^2 a + px = g_1(\mathbf{v}), \text{ and}\\
&g_2(\mathbf{v}') = (6/5 - p)^2 b' = (6/5 - p)^2 b + 2(6/5 - p)^2 y = g_2(\mathbf{v}).
\end{aligned}$$

The last equality holds because $p = 7/10$. So $g_1(\mathbf{v}') = g_2(\mathbf{v}') = g_1(\mathbf{v})$. This means that $\mathbf{v}' \in \mathscr{M}$, with $e' = 0$ and $y' = 0$; and (3) holds by replacing $\mathbf{v}$ with $\mathbf{v}'$.

Now suppose $a = 0$ and $b = 0$. Then $g_1(\mathbf{v}) = p(1 - 2y)/2$ and $g_2(\mathbf{v}) = (6/5 - p)y$. So $g_1(\mathbf{v}) = g_2(\mathbf{v})$ implies $y = (5/12)p$. So

$$g_1(\mathbf{v}) = \frac{p}{2} - \frac{5p^2}{12} = \frac{3}{20} - \frac{1}{60}(3 - 5p)^2 < \frac{9}{50},$$

and the assertion of the lemma holds.

So we may assume $a \neq 0$ and $b = 0$. Then

$$g_1(\mathbf{v}) = p^2 a + p(1 - a - 2y)/2, \text{ and } g_2(\mathbf{v}) = (6/5 - p)y.$$

Now $\mathbf{v}$ must be a critical point of g_1 subject to $g := g_1 - g_2 = 0$, where g_1, g_2, g are considered as functions of a and y. So there exists λ (Lagrange multiplier) such that $\partial f_1/\partial a = \lambda \partial g/\partial a$ and $\partial f_1/\partial y = \lambda \partial g/\partial y$. This gives

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -p = \lambda\left(-p - (6/5 - p)\right).$$

Since $p \in (1/5, 1)$, we have $p = 1/2$ or $\lambda = 1$ from the first equation. Using the second equation, we get $\lambda \neq 1$ (since $p \in (1/5, 1)$). Hence, $p = 1/2$, and so, $g_1(\mathbf{v}) = (1 - 2y)/4$ and $g_2(\mathbf{v}) = 7y/10$. Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have $(1 - 2y)/4 = 7y/10$, and so $y = 5/24$. Hence $g_2(\mathbf{v}) = 7/48 < 9/50$. This completes the proof of (3).

By (2) and (3), $x \neq 0$ and $\mathbf{v} = (a, b, x, 0, 0, p)$. Hence $x = \frac{1-a-b}{2}$,

$$g_1(\mathbf{v}) = p^2 a + p\frac{1 - a - b}{2}, \text{ and } g_2(\mathbf{v}) = (6/5 - p)^2 b.$$

Note that when $b = 0$, we have $g_2(\mathbf{v}) = 0 < 9/50$; and the assertion of the lemma holds. So we may assume

(4) $b \neq 0$.

We consider two cases: $a \neq 0$, and $a = 0$.

Case 1. $a \neq 0$.

Then $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of a and b. So there exists λ such that $\partial g_1/\partial a = \lambda \partial g/\partial a$ and $\partial g_1/\partial b = \lambda \partial g/\partial b$, which gives

$$p^2 - \frac{p}{2} = \lambda\left(p^2 - \frac{p}{2}\right), \text{ and } -\frac{p}{2} = \lambda\left(-\frac{p}{2} - (6/5 - p)^2\right).$$

Since $p \in (1/5, 1)$, we have $\lambda \neq 1$ from the second equation; so $p^2 - \frac{p}{2} = 0$ (from the first equation), which implies $p = \frac{1}{2}$. Define

$$\mathbf{v}' := (a', b', x', y', e', p') = (a + 2x, b, 0, 0, 0, p).$$

Then $a' + b' + 2(x' + y' + e') = a + b + 2x = 1$ and $g_2(\mathbf{v}) = g_2(\mathbf{v}')$. Also, because $p = 1/2$, $g_1(\mathbf{v}') = p^2 a' = p^2 a + 2p^2 x = g_1(\mathbf{v})$. Therefore, $\mathbf{v}' \in \mathscr{M}'$, contradicting (2).

Case 2. $a = 0$.

Then $g_1(\mathbf{v}) = p(1-b)/2$ and $g_2(\mathbf{v}) = (6/5 - p)^2 b$. Since $b \neq 0$ (by (4)) and $p \in (1/5, 1)$, $\mathbf{v}$ is a critical point of g_1 subject to $g := g_1 - g_2 = 0$, all considered as functions of b and p. So there exists λ such that $\partial g_1/\partial b = \lambda \partial g/\partial b$ and $\partial g_1/\partial p = \lambda \partial g/\partial p$, which gives

$$-\frac{p}{2} = \lambda\left(-\frac{p}{2} - (6/5 - p)^2\right) \text{ and } \frac{1-b}{2} = \lambda\left(\frac{1-b}{2} + 2b\,(6/5 - p)\right).$$

Since $p \in (1/5, 1)$, we have $\lambda \neq 0$ (from the first equation). So

$$\frac{p}{2}\left(\frac{1-b}{2} + 2b\,(6/5 - p)\right) = \frac{1-b}{2}\left(\frac{p}{2} + (6/5 - p)^2\right).$$

By a simple calculation, we derive

$$b = \frac{6/5 - p}{6/5 + p}.$$

Since $g_1(\mathbf{v}) = g_2(\mathbf{v})$, we have

$$b = \frac{p/2}{(6/5 - p)^2 + p/2}.$$

Therefore, we have $(6/5 - p)^3 = p^2$.

Note that $h(p) := (6/5 - p)^3 - p^2$ is a decreasing function over $(1/5, 1)$, and a simple calculation shows $h(11/20) < 0$. So $p < 11/20$.

Now note that $g_1(\mathbf{v}) = \frac{p^2}{6/5+p}$ is an increasing function over $(1/5, 1)$. So

$$g_1(\mathbf{v}) = \frac{p^2}{6/5 + p} < \frac{(11/20)^2}{6/5 + 11/20} < \frac{9}{50}.$$

This completes the proof of the lemma. ∎

4 Bounding three quantities

We now prove lemmas that bound three quantities $f_1(p)$, $f_2(p)$ and $f_3(p)$. The key is to choose p so that $f_1(p) = f_2(p) = f_3(p)$. The next lemma says that under certain conditions, we have $f_i(p) \leq 2/5$ for $i = 1, 2, 3$, or we have $f_1(p) = f_2(p) = f_3(p)$.

Lemma 4.1 *Let $b_{ij}, x_i, a_i, c \in \mathbb{R}^+$, $1 \leq i \neq j \leq 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \geq \max\{2x_i, 2x_j\}$, and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1$. For any permutation ijk of $\{1, 2, 3\}$, let*

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c.$$

Then there exists $p_1, p_2, p_3 \in [0, 1]$ with $p_1 + p_2 + p_3 = 1$ such that

(*i*) $f_i \leq \frac{2}{5}$ *for* $i = 1, 2, 3$, *or* *(ii)* $f_1 = f_2 = f_3$ *and* $p_i \in (0,1)$ *for* $i = 1,2,3$.

Proof. Note that f_i may be viewed as a function of $(a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3)$ which is a point in $[0,1]^{13}$. For any permutation ijk of $\{1,2,3\}$, let

$$\alpha_i := b_{jk} + x_j + x_k,\ \ \beta_i := a_j + a_k,\ \text{ and } \gamma_i := \alpha_i + \beta_i + c.$$

Then for $i = 1,2,3$,

$$f_i = (1-p_i)\alpha_i + (1-p_i)^2\beta_i + (1-p_i)^3 c.$$

By symmetry, we may assume that

$$\gamma_1 \leq \gamma_2 \leq \gamma_3.$$

We may further assume that

(1) $\gamma_1 \geq 2/5$

For, suppose (1) fails. Then $\gamma_1 < 2/5$. Let $p_1 = 0$; then $f_1 = \gamma_1 < 2/5$. We wish to apply Lemma 3.1 to show that there exist $p_2, p_3 \in (0,1)$ such that $p_2 + p_3 = 1$ and $f_2 = f_3 \leq 2/5$. Let

$$m = \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + (\alpha_2 + \alpha_3) + c.$$

Let $x = \alpha_2/m$, $y = \alpha_3/m$, $a = \beta_2/m$, $b = \beta_3/m$, $z = (\alpha_2+\alpha_3)/m$, and $e = c/m$. Then $a+b+x+y+z+e = 1$ and $z \geq x+y$. Thus by Lemma 3.1, there exist $p_2, p_3 \in (0,1)$ such that $p_2 + p_3 = 1$ and $f_2/m = f_3/m \leq 1/6$.

Note that

$$m = 2(b_{13} + x_1 + x_3 + b_{12} + x_1 + x_2) + (a_1 + a_2 + a_1 + a_3) + c \leq 2 + 2x_1.$$

Since $b_{ij} \geq \max\{2x_i, 2x_j\}$ for $1 \leq i \neq j \leq 3$, we have $5x_1 \leq x_1 + b_{12} + b_{13} \leq 1$. Hence $x_1 \leq 1/5$, and so $m \leq 12/5$. Therefore, $f_2 = f_3 \leq (12/5)/6 = 2/5$; so (i) holds. Thus we may assume (1).

We now write $f_i(p_i)$ for f_i, considering it as a function of p_i over $[0,1]$ (while fixing the other parameters). Differentiating about p_i, we have $f_i'(p_i) = -\alpha_i - 2(1-p_i)\beta_i - 3(1-p_i)^2 c \leq 0$ and $f_i''(p_i) = 2\beta_i + 6(1-p_i)c \geq 0$. Note from (1) that $f'(p_i) < 0$ unless $p_i = 1$. So

(2) each $f_i(p_i)$ is both decreasing and convex over $[0,1]$.

Because of (2), we approximate $f_i(p_i)$ (for each i) with the line $h_i(p_i)$ through the the points $(0, f_i(0))$ and $(1, f_i(1))$ in the Euclidean plane. Hence $h_i(p_i) = (1-p_i)\gamma_i$. It is also convenient to consider the reflection of $f_3(p_3)$ with respect to the line $p_3 = 1/2$, namely $f_4(p_3) = f_3(1-p_3) = p_3\alpha_3 + p_3^2\beta_3 + p_3^3 c$. Let $h_4(p_3) = \gamma_3 p_3$, which is the reflection of $h_3(p_3)$ with respect to the line $p_3 = 1/2$.

By (2) and by definition, we have

(3) $f_4(p_3)$ is convex and increasing over $[0,1]$; and for $i = 1,2,3,4$, $f_i(p_i) \leq h_i(p_i)$ when $p_i \in [0,1]$.

For each $1 \leq \alpha \leq \gamma_1$ and for $i = 1,2,3,4$ let $p_i(\alpha)$ denote the unique root of $f_i(p_i) = \alpha$ in $[0,1]$, and $q_i(\alpha)$ the unique root of $h_i(q_i) = \alpha$ in $[0,1]$. Note that from (2) and (3), we have

(4) for $\alpha \in [0, \gamma_1]$ and for $i = 1,2,3$, $p_i(\alpha) \leq q_i(\alpha)$, $p_i(\alpha)$ and $q_i(\alpha)$ are decreasing; and $p_4(\alpha)$ and $q_4(\alpha)$ are increasing.

Let (a, b) be the point where f_2 and f_4 intersect, that is, $f_2(a) = f_4(a) = b$; so $p_2(b) = p_4(b) = a$. Let (a', b') be the point where h_2 and h_4 intersect, i.e., $h_2(a') = h_4(a') = b'$. By (2) and (3), we have $b \leq b'$. By solving $h_2(a') = h_4(a') = b'$, we have

$$a' = \frac{\gamma_2}{\gamma_2 + \gamma_3}, \text{ and } b' = \frac{\gamma_2\gamma_3}{\gamma_2 + \gamma_3}.$$

Since $h_3(1 - a') = h_4(a') = b'$, we have $q_3(b') = 1 - q_2(b')$, and so $q_2(b') + q_3(b') = 1$.

We may assume

(5) $b' = \frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \geq \gamma_1$.

For, suppose $b' < \gamma_1$. Then $b < \gamma_1$; so $p_i(b)$ is defined for $i = 1, 2, 3, 4$. Since f_3 and f_4 are reflections through the line $p_3 = 1/2$, $p_3(b) + p_4(b) = 1$. Since $p_2(b) = p_4(b) = a$ and $p_1(b) > 0$, we have $p_1(b)+p_2(b)+p_3(b) = p_1(b)+1 > 1$. Also, $p_1(\gamma_1) = 0$, and $p_2(\gamma_1)+p_3(\gamma_1) \leq q_2(\gamma_1)+q_3(\gamma_1) < q_2(b')+q_3(b') = 1$; so $p_1(\gamma_1)+p_2(\gamma_2)+p_3(\gamma_3) < 1$. Since $p_1(\alpha)+p_2(\alpha)+p_3(\alpha)$ is a decreasing function of α, there exists $\alpha \in (b, \gamma_1)$ such that $p_1(\alpha) + p_2(\alpha) + p_3(\alpha) = 1$, and (ii) holds.

We claim that

(6) $\gamma_1 \leq 1/2$, $1/2 \leq \gamma_2, \gamma_3 \leq 1$, and $c - \sum_{1\leq i<j\leq 3} b_{ij} \geq 0$.

By (5), $\frac{\gamma_2\gamma_3}{\gamma_2+\gamma_3} \geq \gamma_1$. So by Cauchy-Schwarz,

$$\gamma_2 + \gamma_3 \geq \frac{4}{\frac{1}{\gamma_2} + \frac{1}{\gamma_3}} \geq 4\gamma_1.$$

Since

$$\gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{1\leq i<j\leq 3} b_{ij},$$

we have $5\gamma_1 \leq \gamma_1 + \gamma_2 + \gamma_3 = 2 + c - \sum_{i<j} b_{ij}$, and so $\gamma_1 \leq \frac{2}{5} + \frac{c-\sum_{i<j} b_{ij}}{5}$. Therefore, since $\gamma_2 + \gamma_3 \leq 2$,

$$2 + c - \sum_{i<j} b_{ij} = \gamma_1 + \gamma_2 + \gamma_3 \leq 2 + \frac{2}{5} + \frac{c - \sum_{i<j} b_{ij}}{5}.$$

So $c - \sum_{i<j} b_{ij} \leq \frac{1}{2}$, which in turn implies $5\gamma_1 \leq 2 + c - \sum_{i<j} b_{ij} \leq \frac{1}{2} + 2$. Thus, $\gamma_1 \leq \frac{1}{2}$.

By (1), $\gamma_1 \geq 2/5$. So $2 \leq 5\gamma_1 \leq 2 + c - \sum_{i<j} b_{ij}$, which implies $c - \sum_{i<j} b_{ij} \geq 0$. Hence $\gamma_2 + \gamma_3 \geq 2 + (c - \sum_{i<j} b_{ij}) - \gamma_1 \geq 2 - \gamma_1 \geq \frac{3}{2}$ (because $\gamma_1 \leq 1/2$). Since $0 \leq \gamma_i \leq 1$, for $i = 1, 2$, we have $\frac{1}{2} \leq \gamma_2, \gamma_3 \leq 1$, completing the proof of (6).

We claim that

(7) $x_i \leq 1/9$, for $i = 1, 2, 3$.

For, otherwise, $x_i > 1/9$ for some $i \in \{1, 2, 3\}$. Then $b_{ij} > 2/9$ and $b_{ik} > 2/9$, where $\{i, j, k\} = \{1, 2, 3\}$. So by (6), $c \geq \sum_{i<j} b_{ij} > 4/9$. This implies $c + b_{ij} + b_{ik} + x_i > 1$, contradicting our assumption in the lemma. So we have (7).

Now let $p_1 = 1/5$; then $f_1 \leq \frac{4}{5}\gamma_1 \leq 2/5$ (as $\gamma_1 \leq 1/2$ by (6)). We wish to apply Lemma 3.2 to prove the existence of p_2 and p_3 for which $f_2 \leq 2/5$ and $f_3 \leq 2/5$. Let $p = 1 - p_2$ and $6/5 - p = 1 - p_3$. Let

$$m = \beta_2 + \beta_3 + 2(\alpha_2 + \alpha_3 + c),$$

and let $a = \beta_2/m$, $b = \beta_3/m$, $x = \alpha_2/m$, $y = \alpha_3/m$, and $e = c/m$. Then $a+b+2(x+y+e) = 1$,

$$f_2/m = p^2 a + px + p^3 e, \text{ and } f_3/m = (6/5-p)^2 b + (6/5-p)y + (6/5-p)^3 e.$$

Note that $m = 2a_1 + a_2 + a_3 + 2(b_{12} + b_{13} + 2x_1 + x_2 + x_3 + c) \leq 2 + 2x_1 \leq 20/9$ (by (7)). So it follows from (6) that for $i = 2, 3$,

$$\frac{\alpha_i + \beta_i + c}{m} = \frac{\gamma_i}{m} \geq \frac{1/2}{20/9} = \frac{9}{40}.$$

To apply Lemma 3.2, we need to show that there exists $p \in (1/5, 1)$ such that $f_2 = f_3$. To see this, let $g_1(p) = f_2/m$ and $g_2(p) = f_3/m$, considered as functions of p. We note that

$$g_1(1/5) \leq \frac{a+x+e}{5} = \frac{\alpha_2 + \beta_2 + c}{5m} \leq 1/5, \text{ and}$$
$$g_2(1/5) = b + y + e = \frac{\alpha_3 + \beta_3 + c}{m} \geq 9/40.$$

So $g_1(1/5) < g_2(1/5)$. Similarly, we can show $g_1(1) \geq 9/40 > 1/5 \geq g_2(1)$. By (2), $g_1(p)$ is an increasing function, and $g_2(p)$ is a decreasing function. So there exists $p \in (1/5, 1)$ such that $f_2 = f_3$.

We can now apply Lemma 3.2. As a consequence, there exists $p \in (1/5, 1)$ such that

$$g_1 = px + p^2 a + p^3 e \leq 9/50 \text{ and } g_2 = (6/5-p)^2 b + (6/5-p)y + (6/5-p)^3 e \leq 9/50.$$

Since $m \leq 20/9$, we have $f_2 \leq \frac{9}{50} m \leq \frac{2}{5}$ and $f_3 \leq \frac{9}{50} m \leq \frac{2}{5}$. Note that $p_2 = 1 - p$ and $p_3 = p - 1/5$. Since $p \in (1/5, 1)$, we have $p_2, p_3 \in (0, 1)$. Clearly, $p_1 + p_2 + p_3 = 1$. This completes the proof of the lemma. ∎

The next lemma deals with a special case of the case $c = 0$ in Lemma 2.1.

Lemma 4.2 *Let $b_i, y_i \in \mathbb{R}^+$ for $i = 1, 2, 3$ such that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Suppose there exist $q_i \in (0, 1)$, $i = 1, 2, 3$, such that $q_1 + q_2 + q_3 = 2$ and $2y_1 q_1 + b_1 q_1^2 = 2y_2 q_2 + b_2 q_2^2 = 2y_3 q_3 + b_3 q_3^2$. Then for $i = 1, 2, 3$, $2y_i q_i + b_i q_i^2 \leq 2/5$.*

Proof. For convenience, let $f_i := 2y_i q_i + b_i q_i^2$, $i = 1, 2, 3$. Let $\mathscr{D}$ denote the set of all points $(b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3)$ such that $b_i, y_i \in \mathbb{R}^+$ and $q_i \in (0, 1)$ for $i = 1, 2, 3$,

$$\sum_{i=1}^{3}(3y_i + b_i) = 2,$$
$$q_1 + q_2 + q_3 = 2, \text{ and}$$
$$f_1 = f_2 = f_3.$$

So $\mathscr{D}$ is a compact subset of $[0, 2]^3 \times [0, 2/3]^3 \times [0, 1]^3$. Note that $\mathscr{D} \neq \emptyset$ by assumption of the lemma. Let

$$\mathbf{v} := (b_1, b_2, b_3, y_1, y_2, y_3, q_1, q_2, q_3) \in \mathscr{D}$$

such that $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that $f_1(\mathbf{v}) \leq 2/5$.

Since $q_i \in (0,1)$ and $f_1 = f_2 = f_3$, we see that if $f_i = 0$ for some $i \in \{1,2,3\}$ then $b_i = y_i = 0$ for all $i = 1,2,3$, contradicting the condition that $\sum_{i=1}^{3}(3y_i + b_i) = 2$. Hence, we have
(1) for each $i \in \{1,2,3\}$, $b_i > 0$ or $y_i > 0$.

We may assume that
(2) there exists some $i \in \{1,2,3\}$ such that $b_i > 0$.
For, suppose $b_i = 0$ for $i = 1,2,3$. Then $f_i = 2y_iq_i$ and $y_i > 0$ (by (1)) for $i = 1,2,3$, and $y_1 + y_2 + y_3 = 2/3$. Hence, by Cauchy-Schwarz,

$$\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \geq \frac{9}{y_1 + y_2 + y_3} = \frac{27}{2}.$$

Setting $f_1 = f_2 = f_3 = \alpha$, we have $q_i = \alpha/2y_i$ for $i = 1,2,3$. Therefore, since $q_1 + q_2 + q_3 = 2$,

$$\alpha = \frac{4}{\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}} \leq \frac{8}{27} < \frac{2}{5}.$$

We may also assume that
(3) there exists some $j \in \{1,2,3\}$ such that $y_j > 0$.
For, otherwise, $y_1 = y_2 = y_3 = 0$. Then $f_i = b_iq_i^2$ and $b_i > 0$ (by (1)) for $i = 1,2,3$, and $b_1 + b_2 + b_3 = 2$. Setting $f_1 = f_2 = f_2 = \alpha$, we have $q_i = \sqrt{\alpha/b_i}$. Since $q_1 + q_2 + q_3 = 2$, we have (by Cauchy-Schwarz),

$$\alpha = \frac{4}{\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}} + \frac{1}{\sqrt{b_3}}\right)^2} \leq \frac{4}{81}\left(\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3}\right)^2 \leq \frac{4}{9}\frac{b_1 + b_2 + b_3}{3} = \frac{8}{27} < \frac{2}{5}.$$

We may further assume that
(4) there exists some $i \in \{1,2,3\}$ such that $b_iy_i \neq 0$.
Otherwise, we have two cases (by symmetry): $y_1 = y_2 = b_3 = 0$, or $b_1 = b_2 = y_3 = 0$

First, assume $y_1 = y_2 = b_3 = 0$. Then, $b_1 > 0$, $b_2 > 0$, $y_3 > 0$, $b_1 + b_2 + 3y_3 = 2$,

$$f_1 = b_1q_1^2, \; f_2 = b_2q_2^2, \text{ and } f_3 = 2y_3q_3.$$

Setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} + \frac{\alpha}{2y_3} = 2.$$

So

$$\sqrt{\alpha} = \frac{4}{\sqrt{(1/\sqrt{b_1} + 1/\sqrt{b_2})^2 + 4/y_3} + (1/\sqrt{b_1} + 1/\sqrt{b_2})}.$$

Note that

$$\left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}}\right)^2 \geq \frac{4}{\sqrt{b_1b_2}} \geq \frac{8}{b_1 + b_2} = \frac{8}{2 - 3y_3},$$

so

$$\sqrt{\alpha} \leq \frac{4}{\sqrt{\frac{8}{2-3y_3} + \frac{4}{y_3}} + \sqrt{\frac{8}{2-3y_3}}}.$$

By applying Cauchy-Schwarz, we have

$$\sqrt{\frac{8}{2-3y_3}+\frac{4}{y_3}}=\sqrt{\frac{8}{2-3y_3}+\frac{8}{3y_3}+\frac{4}{3y_3}}\geq\sqrt{\frac{32}{2-3y_3+3y_3}+\frac{4}{3y_3}}=\sqrt{16+\frac{4}{3y_3}}.$$

Recall $y_3 \in (0, 2/3)$. If $y_3 \leq 1/3$, then $\sqrt{16+\frac{4}{3y_3}} \geq \sqrt{20}$ and $\sqrt{8/(2-3y_3)} \geq 2$; so $\sqrt{8/(2-3y_3)+4/y_3}+\sqrt{8/(2-3y_3)} \geq \sqrt{20}+2 > 6.4$. If $y_3 \geq 1/3$, then $\sqrt{16+\frac{4}{3y_3}} \geq \sqrt{18}$ and $\sqrt{8/(2-3y_3)} \geq 2\sqrt{2}$; so $\sqrt{8/(2-3y_3)+4/y_3}+\sqrt{8/(2-3y_3)} \geq \sqrt{18}+2\sqrt{2} > 6.4$. Hence $\sqrt{\alpha} \leq 4/6.4 = 1/1.6$; and so $\alpha < 2/5$.

Now assume $b_1 = b_2 = y_3 = 0$. Then $y_1 > 0$, $y_2 > 0$, $b_3 > 0$, $3(y_1 + y_2) + b_3 = 2$,

$$f_1 = 2y_1q_1,\ f_2 = 2y_2q_2, \text{ and } f_3 = b_3q_3^2.$$

Again, setting $\alpha = f_1 = f_2 = f_3$ and using $q_1 + q_2 + q_3 = 2$, we have

$$\frac{\alpha}{2y_1}+\frac{\alpha}{2y_2}+\frac{\sqrt{\alpha}}{\sqrt{b_3}}=2,$$

So

$$\sqrt{\alpha}=\frac{4}{\sqrt{1/b_3+4(1/y_1+1/y_2)}+1/\sqrt{b_3}}.$$

Note that $1/y_1 + 1/y_2 \geq 4/(y_1 + y_2) = 12/(2 - b_3)$. So

$$\sqrt{\alpha}\leq\frac{4}{\sqrt{1/b_3+48/(2-b_3)}+1/\sqrt{b_3}}.$$

By Cauchy-Schwarz, $1/b_3 + 1/(2 - b_3) \geq 4/(b_3 + 2 - b_3) = 2$. So

$$h(b_3):=\sqrt{2+47/(2-b_3)}+1/\sqrt{b_3}\leq\sqrt{1/b_3+48/(2-b_3)}+1/\sqrt{b_3}.$$

Recall that $b_3 \in (0, 2)$. If $b_3 \geq 1/2$ then $h(b_3) \geq \sqrt{2+47/(2-1/2)}+1/\sqrt{2} > \sqrt{33}+1/\sqrt{2} > 6.4$; if $b_3 \leq 1/2$ then $h(b_3) \geq \sqrt{2+47/2}+\sqrt{2} > 6.4$. So $\sqrt{\alpha} < 4/6.4 = 1/1.6$; and hence $\alpha < 2/5$.

By (4) and by symmetry, we may assume that
(5) $b_3y_3 \neq 0$.

We may further assume that
(6) $b_1y_1 = 0$ and $b_2y_2 = 0$.
For, otherwise, by symmetry, assume $b_2y_2 > 0$. Then $\mathbf{v}$ is a solution to the following optimization problem:

Maximize f_1
subject to

$$\begin{aligned}
h_1 &:= f_1 - f_2 = 0,\\
h_2 &:= f_1 - f_3 = 0,\\
h_3 &:= 3(y_1 + y_2 + y_3) + (b_1 + b_2 + b_3) - 2 = 0,\\
h_4 &:= q_1 + q_2 + q_3 - 2 = 0.
\end{aligned}$$

Applying the method of Lagrange multipliers, we have, for $u \in \{y_i, b_i : i = 2, 3\}$,

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

Thus,

$$\begin{aligned}
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3,\\
&\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3.
\end{aligned}$$

Clearly, if $\lambda_i = 0$ for some $i \in \{1, 2, 3\}$ then $\lambda_i = 0$ for all $i = 1, 2, 3$ (since $q_i \in (0, 1)$). In fact, $\lambda_i \neq 0$ for all $i = 1, 2, 3$. To see this we notice that either $b_1 > 0$ or $y_1 > 0$, so $\mathbf{v}$ also satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u = b_1$ or $u = y_1$. For $u = b_1$, we have $q_1^2 = \lambda_1 q_1^2 + \lambda_2 q_1^2 + \lambda_3$, and for $u = y_1$ we have $2q_1 = \lambda_1 2q_1 + \lambda_2 2q_1 + 3\lambda_3$. In either case, we see that $\lambda_i \neq 0$ (since $q_1 \neq 0$).

Now using the partial derivatives about b_2 and y_2, we get $q_2 = 2/3$; and using the partial derivatives about b_3 and y_3 we obtain $q_3 = 2/3$. So $q_1 = 2/3$ since $q_1 + q_2 + q_3 = 2$. Then for $i = 1, 2, 3$,

$$f_i = \frac{4}{3} y_i + \frac{4}{9} b_i = \frac{4}{9}(3y_i + b_i).$$

Since $f_1 = f_2 = f_3$ and $\sum_{i=1}^{3}(3y_i + b_i) = 2$, we get $3y_i + b_i = \frac{2}{3}$ for $i = 1, 2, 3$, and hence $f_i = \frac{8}{27} < \frac{2}{5}$.

By (5) and (6), we have three cases to consider: $b_1 = b_2 = 0$; $y_1 = y_2 = 0$; $y_1 = b_2 = 0$ or $b_1 = y_2 = 0$. Let h_1, h_2, h_3, h_4 be defined as in the proof of (6).

Case 1. $b_1 = b_2 = 0$.

Then $y_1 > 0$, $y_2 > 0$, $f_1 = 2y_1 q_1$, $f_2 = 2y_2 q_2$, $f_3 = 2y_3 q_3 + b_3 q_3^2$. Moreover, $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of $y_1, y_2, y_3, b_3, q_1, q_2, q_3$. Hence for $u \in \{y_1, y_2, y_3, b_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

So

$$\begin{aligned}
&\text{for } u = y_1, \text{ we have } 2q_1 = \lambda_1(2q_1) + \lambda_2(2q_1) + 3\lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives about b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Set $\alpha := 2y_1 q_1 = 2y_2 q_2 = 4(3y_3 + b_3)/9$. In particular, $\alpha = 4(3y_3 + b_3)/9 = 4(2 - 3(y_1 + y_2))/9$, and so $y_1 + y_2 = 2/3 - 3\alpha/4$. Using $q_1 + q_2 = 4/3$ and Cauchy-Schwarz, we get

$$\frac{4}{3} = \frac{\alpha}{2y_1} + \frac{\alpha}{2y_2} \geq \frac{2\alpha}{y_1 + y_2} = \frac{2\alpha}{2/3 - 3\alpha/4}.$$

This implies $\alpha \leq 8/27 < 2/5$.

Case 2. $y_1 = y_2 = 0$.

Then $b_1 > 0$, $b_2 > 0$, $f_1 = b_1 q_1^2$, $f_2 = b_2 q_2^2$ and $f_3 = 2y_3 q_3 + b_3 q_3^2$. Now $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of $b_1, b_2, y_3, b_3, q_1, q_2, q_3$. Hence for $u \in \{b_1, b_2, b_3, y_3\}$, $\mathbf{v}$ satisfies

$$\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u.$$

So

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = b_2, \text{ we have } 0 = \lambda_1(-q_2^2) + \lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives about b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$. Setting $\alpha := y_1 q_1^2 = y_2 q_2^2 = 4(3y_3 + b_3)/9$, we have $q_i = \sqrt{\alpha}/\sqrt{b_i}$ for $i = 1, 2$, $3y_3 + b_3 = 9\alpha/4$, and $b_1 + b_2 = 2 - 9\alpha/4$. So

$$\frac{4}{3} = \frac{\sqrt{\alpha}}{\sqrt{b_1}} + \frac{\sqrt{\alpha}}{\sqrt{b_2}} \geq \frac{2\sqrt{\alpha}}{\sqrt{\sqrt{b_1}\sqrt{b_2}}} \geq \frac{2\sqrt{2\alpha}}{\sqrt{b_1 + b_2}} = \frac{2\sqrt{2\alpha}}{\sqrt{2 - 9\alpha/4}}.$$

This gives $\alpha \leq 8/27 < 2/5$.

Case 3. $y_1 = b_2 = 0$, or $y_2 = b_1 = 0$.

By symmetry, we may assume that $y_1 = b_2 = 0$. Then $b_1 > 0$, $y_2 > 0$, $b_1 + 3y_2 + (3y_3 + b_3) = 2$, $f_1 = b_1 q_1^2$, $f_2 = 2y_2 q_2$, and $f_3 = 2y_3 q_3 + b_3 q_3^2$.

So $\mathbf{v}$ is a critical point of f_1 subject to $h_1 = h_2 = h_3 = h_4 = 0$, all considered as functions of $b_1, y_2, b_3, y_3, q_1, q_2, q_3$. Hence $\mathbf{v}$ satisfies $\partial f_1/\partial u = \lambda_1 \partial h_1/\partial u + \lambda_2 \partial h_2/\partial u + \lambda_3 \partial h_3/\partial u + \lambda_4 \partial h_4/\partial u$ for $u \in \{b_1, y_2, b_3, y_3\}$. So

$$\begin{aligned}
&\text{for } u = b_1, \text{ we have } q_1^2 = \lambda_1(q_1^2) + \lambda_2(q_1^2) + \lambda_3,\\
&\text{for } u = y_2, \text{ we have } 0 = \lambda_1(-2q_2) + 3\lambda_3,\\
&\text{for } u = b_3, \text{ we have } 0 = \lambda_2(-q_3^2) + \lambda_3\\
&\text{for } u = y_3, \text{ we have } 0 = \lambda_2(-2q_3) + 3\lambda_3.
\end{aligned}$$

Clearly, $\lambda_i \neq 0$ for $i = 1, 2, 3$. So from the partial derivatives about b_3 and y_3, we have $q_3 = 2/3$, and hence $q_1 + q_2 = 4/3$.

Suppose for contradiction that $f_i > 2/5$. Then

$$b_1 > \frac{2}{5q_1^2}, \quad y_2 > \frac{1}{5q_2}, \text{ and } 3y_3 + b_3 > \frac{9}{10}.$$

Hence,

$$2 = b_1 + 3y_2 + (3y_3 + b_3) > \frac{2}{5q_1^2} + \frac{3}{5q_2} + \frac{9}{10}.$$

Since $q_1 + q_2 = 4/3$, we have $q_1 \le 2/3$ or $q_2 \le 2/3$. Because $q_1, q_2 \in (0,1)$, we see that

$$\frac{2}{5q_1^2} + \frac{3}{5q_2} > \frac{2}{5} + \frac{3}{5\cdot(2/3)} = \frac{13}{10}, \text{ or } \frac{2}{5q_1^2} + \frac{3}{5q_2} > \frac{2}{5(2/3)^2} + \frac{3}{5} = \frac{3}{2}.$$

But this implies

$$2 > \frac{2}{5q_1^2} + \frac{3}{5q_2} + \frac{9}{10} > \frac{13}{10} + \frac{9}{10},$$

a contradiction. ∎

We are now ready to prove a lemma which will be used to deal with the case $c = 0$ in Lemma 2.1.

Lemma 4.3 *Let $a_i, x_i, b_{ij} \in \mathbb{R}^+$, $1 \le i \ne j \le 3$, such that $b_{ij} = b_{ji}$, $b_{ij} \ge \max\{2x_i, 2x_j\}$ and $b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 1$. For any permutation ijk of $\{1,2,3\}$, let*

$$f_k := (1-p_k)(b_{ij} + x_i + x_j) + (1-p_k)^2(a_i + a_j)$$

Suppose there exist $p_1, p_2, p_3 \in (0,1)$ such that $p_1 + p_2 + p_3 = 1$ and $f_1 = f_2 = f_3$. Then for $k = 1,2,3$, $f_k \le 2/5$.

Proof. For any permutation ijk of $\{1,2,3\}$, and let $y_k = x_i + x_j$ and $b_k = a_i + a_j$. Then

$$f_k = (1-p_k)(b_{ij} + y_k) + (1-p_k)^2 b_k.$$

Set $\alpha = f_1 = f_2 = f_3$. Note that we may assume $\alpha > 0$ (otherwise we are done); and hence $b_{ij} + y_k + b_k > 0$ for $k = 1,2,3$. Since $p_k \in (0,1)$, $1 - p_k \in (0,1)$; and hence by solving $f_k(p_k) = \alpha$ we get

$$1 - p_k = \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)}.$$

We wish to show that $\alpha \le \frac{2}{5}$. So we consider the following optimization problem.

Maximize α

Subject to

$$g_1 := \sum_{k=1}^{3} \frac{2\alpha}{\sqrt{(b_{ij} + y_k)^2 + 4b_k\alpha} + (b_{ij} + y_k)} - 2 = 0,$$

$$g_2 := b_{12} + b_{13} + b_{23} + \frac{1}{2}(y_1 + y_2 + y_3 + b_1 + b_2 + b_3) - 1 = 0,$$

$$b_{ij} \ge y_k \ge 0, \text{ for } \{i,j,k\} = \{1,2,3\}.$$

Here, g_1, g_2 are considered as functions of α, b_{ij}, b_k, y_k. By the assumption of the lemma, the feasible region of this optimization problem is nonempty. It suffices to show that the maximum α of this problem is at most $2/5$.

Claim 1. α is maximized only when $b_{ij} = y_k$ or $y_k = 0$, for all $\{i,j,k\} = \{1,2,3\}$.

For, suppose $b_{ij} > y_k > 0$ for some permutation ijk of $\{1,2,3\}$. By applying the method of Lagrange multipliers, we have $\partial\alpha/\partial u = \lambda_1 \partial g_1/\partial u + \lambda_2 \partial g_2/\partial u$, where $u \in \{\alpha, b_{ij}, y_k\}$. So

$$\text{for } u = b_{ij}, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij}+y_k)^2+4b_k\alpha}\left(\sqrt{(b_{ij}+y_k)^2+4b_k\alpha}+(b_{ij}+y_k)\right)^2} + \lambda_2,$$

$$\text{for } u = y_k, \quad 0 = \lambda_1 \frac{-2\alpha\left(b_{ij} + y_k + \sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha}\right)}{\sqrt{(b_{ij}+y_k)^2+4b_k\alpha}\left(\sqrt{(b_{ij}+y_k)^2+4b_k\alpha}+(b_{ij}+y_k)\right)^2} + \frac{\lambda_2}{2},$$

$$\text{for } u = \alpha, \quad 1 = \lambda_1 \frac{\partial g_1}{\partial \alpha} + \lambda_2 \frac{\partial g_2}{\partial \alpha}.$$

The first two equations give $\lambda_1 = \lambda_2 = 0$, which contradicts the third equation.

Therefore, the maximum of α is achieved when $b_{ij} = y_k$ for some permutation ijk of $\{1,2,3\}$, or when $y_k = 0$ for some $k \in \{1,2,3\}$; so Claim 1 follows.

Claim 2. We may assume that α is maximized when $b_{ij} > y_k$ for some $\{i,j,k\} = \{1,2,3\}$.

For otherwise, the maximum of α is achieved when $b_{ij} = y_k$ for all permutations ijk of $\{1,2,3\}$. Set $q_k = 1-p_k$ for $k = 1,2,3$; and so $f_k = 2y_kq_k + b_kq_k^2$ and $3(y_1+y_2+y_3)+b_1+b_2+b_3 = 2$. We can now apply Lemma 4.2 and conclude that $f_k \le 2/5$ for $k = 1,2,3$. So Claim 2 holds.

From Claim 1 and Claim 2, we deduce

Claim 3. α is maximized when there exists a permutation ijk of $\{1,2,3\}$ such that $b_{ij} > 0$ and $y_k = 0$ (and so $x_i = x_j = 0$).

We consider three cases.

Case 1. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k = 0$.

Then $b_{ij} + a_k = 1$, $f_k = (1-p_k)b_{ij}$, $f_i = (1-p_i)^2 a_k$, and $f_j = (1-p_j)^2 a_k$.

Since $f_i = f_j$, we have $p_i = p_j$. In particular, $p_i \in (0, 1/2)$ as $p_i + p_j + p_k = 1$. Since $b_{ij} = 1 - a_k$ and $f_k = f_i$, we have $2p_i(1-a_k) = (1-p_i)^2 a_k$. Therefore, $a_k = 2p_i/(1+p_i^2)$, and so,

$$\alpha = \frac{2p_i(1-p_i)^2}{1+p_i^2} = \frac{4}{1+p_i^2} + 2p_i - 4.$$

Suppose, for a contradiction, that $\alpha > 2/5$. Then $\frac{2}{1+p_i^2} + p_i - 2 > 1/5$. Hence for $p_i \in (0, 1/2)$,

$$g(p_i) := 5p_i^3 - 11p_i^2 + 5p_i - 1 > 0.$$

Differentiating about p_i, we get

$$g'(p_i) = 15p_i^2 - 22p_i + 5, \text{ and } g''(p_i) = 30p_i - 22.$$

Note that $g''(p_i) < 0$ for $p_i \in [0, 1/2]$. So $g(p_i)$ has a local maximum when $g'(q_i) = 0$.

Solving $g'(p_i) = 0$, we get $p_i = (11 - \sqrt{46})/15$. Note that $g(0) = -1 < 0$ and $g(1/2) = 5/8 - 11/4 + 5/2 - 1 < 0$. A straightforward calculation shows that $g((11-\sqrt{46})/15) = (82\sqrt{46} - 752)/675 < 0$. So $g(p_i) < 0$ for $p_i \in [0, 1/2]$, a contradiction. Hence $\alpha \le 2/5$.

Case 2. α is maximized when $x_k = b_{ik} = b_{jk} = 0$ and $b_k > 0$.

Then $f_i = (1-p_i)^2 b_i$, and $f_j = (1-p_j)^2 b_j$ and $f_k = (1-p_k)b_{ij} + (1-p_k)^2 b_k$. From $\partial\alpha/\partial b_k = \lambda_1 \partial g_1/\partial b_k + \lambda_2 \partial g_2/\partial b_k$, we obtain

$$0 = \lambda_1 \frac{-4\alpha^2}{\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha}\left(\sqrt{(b_{ij}+y_k)^2 + 4b_k\alpha} + (b_{ij}+y_k)\right)^2} + \frac{\lambda_2}{2}.$$

Using this and the partial derivatives about $u \in \{\alpha, b_{ij}\}$ (as in the proof of Claim 1), we deduce that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and

$$4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}.$$

Therefore, for this case, α is maximized when $4\alpha = b_{ij} + \sqrt{b_{ij}^2 + 4b_k\alpha}$, which implies $p_k = 1/2$.

Write $b'_k := b_k + 2b_{ij}$; then $f_k = (1-p_k)^2 b'_k$ (because $p_k = 1/2$). Note that $(b'_k + b_i + b_j)/2 = b_{ij} + (b_k + b_i + b_j)/2 = 1$. Since $\alpha = f_1 = f_2 = f_3$ and $(1-p_i) + (1-p_j) + (1-p_k) = 2$, we have

$$\frac{\sqrt{\alpha}}{\sqrt{b'_k}} + \frac{\sqrt{\alpha}}{\sqrt{b_i}} + \frac{\sqrt{\alpha}}{\sqrt{b_j}} = 2.$$

Applying Cauchy-Schwarz, we have

$$\alpha = \left(\frac{2}{\frac{1}{\sqrt{b'_k}} + \frac{1}{\sqrt{b_i}} + \frac{1}{\sqrt{b_j}}}\right)^2 \leq 4\left(\frac{\sqrt{b'_k} + \sqrt{b_i} + \sqrt{b_j}}{9}\right)^2 \leq \frac{4}{9}\frac{b'_k + b_i + b_j}{3} = \frac{8}{27} < \frac{2}{5}.$$

Case 3. α is maximized when (i) $x_k > 0$, or (ii) $x_k = 0$ and $b_{ik} > 0$ or $b_{jk} > 0$.

We claim that there exist $a'_m, x'_m, b'_{mn} \in \mathbb{R}^+$, $1 \leq m \neq n \leq 3$, such that $b'_{mn} = b'_{nm}$,

$$\begin{aligned}
&b'_{mn} \geq \max\{2x'_m, 2x'_n\},\\
&b'_{12} + b'_{23} + b'_{31} + x'_1 + x'_2 + x'_3 + a'_1 + a'_2 + a'_3 = 1,\\
&b'_{mn} + x'_m + x'_n \geq b_{mn} + x_m + x_n,\\
&a'_m + a'_n = a_m + a_n, \text{ and}\\
&b'_{st} + x'_s + x'_t > b_{st} + x_s + x_t \text{ for some } 1 \leq s \neq t \leq 3.
\end{aligned}$$

There are two cases to consider. First, suppose $x_k > 0$. Then there exists $\delta > 0$ such that $x'_k = x_k - \delta > 0$ and $b'_{ij} = b_{ij} - 2\delta \geq 2\delta$. Let $b'_{ik} = b_{ik} + \delta$ and $x'_i = x'_j = \delta$. In particular, $x_k > \delta$; and so $b_{ik} \geq 2x_k \geq 2\delta$ and $b_{jk} \geq 2x_k \geq 2\delta$. It is easy to verify that the claim holds by setting $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$. Now assume that $x_k = 0$, and $b_{ik} > 0$ or $b_{jk} > 0$. We may assume $b_{ik} > 0$; the case $b_{jk} > 0$ is symmetric. Then there exists $\delta > 0$ such that $b'_{ik} = b_{ik} - \delta/2 \geq \delta$ and $b'_{ij} = b_{ij} - \delta/2 \geq \delta$. Let $b'_{jk} = b_{jk} + \delta/2$ and $x'_i = \delta/2$. It is easy to verify that the claim holds by setting $x'_j = x_j = 0$, $x'_k = x_k = 0$, $a'_i = a_i$, $a'_j = a_j$ and $a'_k = a_k$.

For every permutation mnl of $\{1,2,3\}$, let

$$f'_l := (1-p_l)(b'_{mn} + x'_m + x'_n) + (1-p_l)^2(a'_m + a'_n).$$

For convenience of comparison, recall that

$$f_l = (1-p_l)(b_{mn} + x_m + x_n) + (1-p_l)^2(a_m + a_n).$$

By Lemma 4.1, there exist $p'_i \in [0,1]$ with $p'_1 + p'_2 + p'_3 = 1$ such that $f'_l(p'_l) \le 2/5$ for $l = 1,2,3$, or $f'_1(p'_1) = f'_2(p'_2) = f'_3(p'_3)$ and $p'_i \in (0,1)$. Since $p_i \in [0,1]$ and $p_1 + p_2 + p_3 = 1$, there exists some l such that $1 - p_l \le 1 - p'_l$.

If $f'_i(p'_i) \le 2/5$ for $i = 1,2,3$ then, since $b'_{mn} + x'_m + x'_n \ge b_{mn} + x_m + x_n$ and $a'_m + a'_n = a_m + a_n$ for all $\{m,n,l\} = \{1,2,3\}$, we have $f_l(p_l) \le f'_l(p'_l) \le 2/5$. Hence $\alpha \le 2/5$.

So we may assume $f'_1(p'_1) = f'_2(p'_2) = f'_3(p'_3)$. Suppose $1 - p_l < 1 - p'_l$. Then, since $b'_{mn} + x'_m + x'_n \ge b_{mn} + x_m + x_n$ and $a'_m + a'_n = a_m + a_n$, and because $b_{mn} + x_m + x_n + a_m + a_n > 0$ (see the begining of the proof), we have $f_l(p_l) < f'_l(p'_l)$, contradicting the maximality of α. So $1 - p_l = 1 - p'_l$. Then $(1 - p'_m) + (1 - p'_n) = (1 - p_m) + (1 - p_n)$. So we may assume that $1 - p_n \le 1 - p'_n$. By the same argument above for $1 - p'_l = 1 - p_l$, we derive the contradiction $f_n(p_n) < f'_n(p'_n)$ if $1 - p_n < 1 - p'_n$; and so we must have $1 - p'_n = 1 - p_n$. Hence we have $p'_i = p_i$ for $i = 1,2,3$. Recall that there exist $1 \le s \ne t \le 3$ such that $b'_{st} + x'_s + x'_t > b_{st} + x_s + x_t$. Let $r \in \{1,2,3\} \setminus \{s,t\}$. Then $f_r(p_r) < f'_r(p'_r)$, again a contradiction to the maximality of α. ∎

5 Conclusion

Having proved the all necessary lemmas, we can now complete the proof of Lemma 2.1. For $i = 1,2,3$, let

$$f_i := (1 - p_i)(b_{jk} + x_j + x_k) + (1 - p_i)^2(a_j + a_k) + (1 - p_i)^3 c.$$

By Lemma 4.1, we may assume that there exist $p_1, p_2, p_3 \in (0,1)$ with $p_1 + p_2 + p_3 = 1$ such that $f_1 = f_2 = f_3$. Let $\mathscr{D}$ denote the set of points $(a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in [0,1]^{13}$ satisfying

$$\begin{aligned}
&b_{ij} \ge \max\{2x_i, 2x_j\},\\
&b_{12} + b_{23} + b_{31} + x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + c = 1,\\
&p_1 + p_2 + p_3 = 1,\\
&p_i \in (0,1) \text{ for } i = 1,2,3, \text{ and}\\
&f_1 = f_2 = f_3.
\end{aligned}$$

Note that $\mathscr{D}$ is a compact subset of $[0,1]^{13}$. So $f_1(\mathbf{v})$ has an absolute maximum over $\mathscr{D}$. Let $\mathscr{M}$ denote all $\mathbf{v} \in \mathscr{D}$ for which $f_1(\mathbf{v})$ is the maximum of f_1 over $\mathscr{D}$. It suffices to show that there is some $\mathbf{v} \in \mathscr{M}$ such that $f_i(\mathbf{v}) \le \frac{2}{5}$ for $i = 1,2,3$. Let

$$\mathbf{v} := (a_1, a_2, a_3, x_1, x_2, x_3, b_{12}, b_{23}, b_{31}, c, p_1, p_2, p_3) \in \mathscr{M}.$$

We claim that $\mathbf{v}$ may be chosen so that $c = 0$. For, suppose $c \ne 0$. Define

$$\mathbf{v}' := (a'_1, a'_2, a'_3, x'_1, x'_2, x'_3, b'_{12}, b'_{23}, b'_{31}, c', p'_1, p'_2, p'_3)$$

such that

- $c' = 0$,
- for $i = 1,2,3$, $a'_i = a_i + p_i c$, $x'_i = x_i$ and $p'_i = p_i$, and

- $b'_{ij} = b_{ij}$ for $1 \leq i \neq j \leq 3$.

Then it is easy to see that $f_i(\mathbf{v}') = f_i(\mathbf{v})$ for $i = 1, 2, 3$, $b'_{ij} \geq \max\{2x'_i, 2x'_j\}$ for $1 \leq i \neq j \leq 3$, and

$$a'_1 + a'_2 + a'_3 + x'_1 + x'_2 + x'_3 + b'_{12} + b'_{13} + b'_{23} + c' = 1.$$

So $\mathbf{v}' \in \mathscr{M}$ is as desired.

Now it follows from Lemma 4.3 that for any $i = 1, 2, 3$, $f_i(\mathbf{v}) \leq \frac{2}{5}$. ∎

One of the reasons that our proof of Lemma 2.1 does not give a better bound than 2/5 is Lemma 4.1. When $\alpha < 2/5$, we could not guarantee the existence of $p_i \in (0, 1)$ such that $p_1 + p_2 + p_3 = 1$ and $f_i \leq \alpha$. Any improvement on this bound should lead to a better bound in Theorem 1.2, which would then imply that Conjecture 1.1 holds for $r = 3$ when m is sufficiently large.

We also mention a related problem for graphs. It is conjectured in [7] that every graph with m edges admits a k-partition, $k \geq 3$, such that $d(V_i) \geq 2m/(2k-1)$. The complete graph on $2k-1$ vertices shows that the lower bound is best possible. This conjecture is shown to be true in [14] for sufficiently large m. In fact, it is shown [14] that $d(V_i) \geq m/(k-1) + o(m)$. It may be possible to demand $d(V_i) \geq (2k-1)m/k^2 + o(m)$.

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