On a coloring conjecture of Hajós

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Abstract

Hajós conjectured that graphs containing no subdivision of K_5 are 4-colorable. It is shown in [15] that if there is a counterexample to this conjecture then any minimum such counterexample must be 4-connected. In this paper, we further show that if G is a minimum counterexample to Hajós' conjecture and S is a 4-cut in G then $G - S$ has exactly two components.

AMS Subject Classification: 05C15, 05C38, 05C40, 05C75
Keywords: Coloring, subdivision of graph, independent paths, planar graph

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1 Introduction

All graphs considered in this paper are simple. The well known Kuratowski's theorem states that a graph is planar iff it contains no subdivision of K_5 or $K_{3,3}$. Thus the *Four Color Theorem* can be stated as follows: Any graph containing no subdivision of K_5 or $K_{3,3}$ is 4-colorable. It is not hard to show that any 3-connected nonplanar graph other than K_5 contains a subdivision of $K_{3,3}$. A straightforward procedure then determines when a graph without containing a subdivision of $K_{3,3}$ is 5-colorable or 4-colorable.

Therefore, it is natural to ask whether graphs containing no subdivision of K_5 are 4-colorable. In fact, Hajós (see [1]) conjectured that graphs containing no subdivision of K_{k+1} are k-colorable, which is false for $k \geq 6$ as shown in [1] (actually it fails for almost all graphs, see Erdös and Fajtlowicz [3]). However, Hajós' conjecture is true for $k = 1, 2, 3$, and remains open for the cases $k = 4$ and $k = 5$.

In this paper, we consider the $k = 4$ case of this conjecture that graphs containing no subdivision of K_5 are 4-colorable (which will be referred to simply as Hajós' conjecture). Thus, it is desirable to understand the structure of graphs containing no subdivision of K_5. Mader [7] proved that every graph on n vertices with at least $3n - 5$ edges contains a subdivision of K_5, establishing a conjecture of Dirac [2]. Seymour [10] in 1977 and, independently, Kelmans [4] in 1979 conjectured that every 5-connected non-planar graph contains a subdivision of K_5. Thus, if a counterexample to Hajós' conjecture is 5-connected then, by the Four Color Theorem, the Kelmans-Seymour conjecture implies Hajós' conjecture. Recently, the Kelmans-Seymour conjecture was proved [5,6] for graphs containing K_4^-, the graph obtained from K_4 by removing an edge.

In [15], it is shown that any minimum counterexample to Hajós' conjecture must be 4-connected. By a minimum counterexample we mean a counterexample with the minimum number of vertices. In this paper, we provide further information about such a counterexample.

Theorem 1.1 *Suppose G is a minimum counterexample to Hajós' conjecture and S is a 4-cut in G. Then $G - S$ has exactly two components.*

Theorem 1.1 is proved in Section 3, and the lemmas used in its proof are given in Section 2. It would be desirable to strengthen Theorem 1.1 such that one of the two components of $G - S$ is trivial (so that any minimum counterexample to Hajós' conjecture is internally 5-connected).

We conclude this section with some notation and terminology. A *separation* in a graph G consists of a pair of subgraphs G_1, G_2 of G, denoted as (G_1, G_2), such that $E(G_1 \cap G_2) = \emptyset$ and for $i = 1, 2$, $E(G_i) \neq \emptyset$ or $V(G_i) - V(G_1 \cap G_2) \neq \emptyset$. The *order* of this separation is $|V(G_1 \cap G_2)|$, and (G_1, G_2) is said to be a *k-separation* if its order is k. Thus, a set $S \subseteq V(G)$ is a k-cut in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$ and $V(G_i) - S \neq \emptyset$ for $i \in \{1, 2\}$. If $v \in V(G)$ and $\{v\}$ is a 1-cut of G, then v is said to be a *cut vertex* of G.

The *ends* of a path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A path P with ends u and v is also said to be *from u to v* or *between u and v*. For a path P and $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y. A collection of paths are said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection.

Let K be a subdivision of K_5. The vertices of degree 4 in K are called the *branch vertices* of K, and the *branch set* of K is simply the set of its branch vertices. Thus, a subdivision of K_5 is the union of 10 independent paths joining its branch vertices.

2 Lemmas

The following result is proved in [15] by Yu and Zickfeld.

Lemma 2.1 (Yu and Zickfeld) *Suppose G is a graph that contains no subdivision of K_5, is not 4-colorable, and subject to these conditions, $|G|$ is minimum. Then G is 4-connected.*

This lemma is proved in [15] using the following result of Watkins and Mesner [14] on cycles through three vertices. See Figure 1 for an illustration.

Lemma 2.2 (Watkins and Mesner) *Let H be a 2-connected graph and let v_1, v_2, v_3 be three distinct vertices of H. Then H has no cycle containing $\{v_1, v_2, v_3\}$ iff one of the following statements holds.*

(i) There exists a 2-cut T in H and, for $u \in \{v_1, v_2, v_3\}$, there exist pairwise disjoint subgraphs D_u of $H - T$ such that $u \in V(D_u)$ and each D_u is a union of components of $H - T$.

(ii) For $u \in \{v_1, v_2, v_3\}$, there exist 2-cuts T_u of H and pairwise disjoint subgraphs D_u of H, such that $u \in V(D_u)$, each D_u is a union of components of $H - T_u$, $T_{v_1} \cap T_{v_2} \cap T_{v_3} = \{a_1\}$, and $T_{v_1} - \{a_1\}, T_{v_2} - \{a_1\}, T_{v_3} - \{a_1\}$ are pairwise disjoint.

(iii) For $u \in \{v_1, v_2, v_3\}$, there exist pairwise disjoint 2-cuts T_u in H and pairwise disjoint subgraphs D_u of $H - T_u$ such that $u \in V(D_u)$, D_u is a union of components of $H - T_u$, and $H - V(D_{v_1} \cup D_{v_2} \cup D_{v_3})$ has precisely two components, each containing exactly one vertex from T_u for $u \in \{v_1, v_2, v_3\}$.

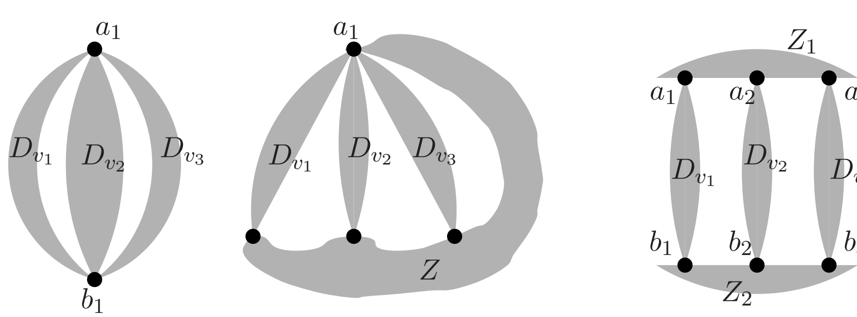

Figure 1: The subgraphs D_u in H, $u \in \{v_1, v_2, v_3\}$.

We need the following result about disjoint paths between two given pairs of vertices, which is a direct consequence of a more general result of Seymour [11] (equivalent versions can be found in [9,12,13]). To state this result, we introduce the following definition. Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. We say that G is $(4, \{s_1, s_2, t_1, t_2\})$-*connected* if, for any k-cut S in G with $k \leq 3$, every component of $G - S$ contains a vertex from $\{s_1, s_2, t_1, t_2\}$. We say that (G, s_1, s_2, t_1, t_2) is *planar* if G can be drawn in a closed disc with no edge-crossings and s_1, s_2, t_1, t_2 occur on the boundary of the disc in this cyclic order.

Lemma 2.3 (Seymour) *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G, and assume that G is $(4, \{s_1, s_2, t_1, t_2\})$-connected. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is planar.*

We will also use the $k = 1$ case of the following result of Perfect [8].

Lemma 2.4 (Perfect) *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G - u)$. Suppose there exist k independent paths from u to distinct $a_1, \ldots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \ldots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \ldots, P_n$ may be chosen so that $a_i \in V(P_i)$ for $i = 1, \ldots, k$.*

3 Proof of Theorem 1.1

By assumption of Theorem 1.1, G contains no subdivision of K_5, G is not 4-colorable, and, subject to these conditions, $|G|$ is minimum. By Lemma 2.1, G is 4-connected.

Suppose $S = \{v_1, v_2, v_3, v_4\}$ is a 4-cut in G, and $C_1, \ldots, C_k$ are the components of $G - S$, with $k \geq 3$. For $i = 1, \ldots, k$, let $x_i \in V(C_i)$ and $H_i := G[V(C_i) \cup S]$ (the subgraph of G induced by $V(C_i) \cup S$). By Menger's theorem, H_i contains four independent paths P_j^i from x_i to v_j, $j = 1, 2, 3, 4$, respectively.

(1) For any $1 \leq s, t \leq k$ (with $s = t$ when $k = 3$) and for any $1 \leq r \leq 4$, $H_s \cup H_t - v_r$ has no cycle containing $S - \{v_r\}$.

For, suppose C is a cycle in $H_s \cup H_t - v_r$ containing $S - \{v_r\}$. Let $p, q \in \{1, \ldots, k\} - \{s, t\}$ be distinct (which exist as $k \geq 3$, and $s = t$ when $k = 3$). Now the paths P_j^p, P_j^q, $j = 1, 2, 3, 4$, and the cycle C form a subdivision of K_5 in G with branch set $(S - \{v_r\}) \cup \{x_p, x_q\}$, a contradiction. This proves (1).

Next we show that

(2) $k = 3$.

Suppose to the contrary that $k \geq 4$. Then S is an independent set in G; for, otherwise, assume $v_1 v_2 \in E(G)$, then $P_1^3 \cup P_3^3 \cup P_2^4 \cup P_3^4$ and $v_1 v_2$ form a cycle in $H_3 \cup H_4 - v_4$ containing $\{v_1, v_2, v_3\}$, which contradicts (1).

Moreover, for $1 \leq i \leq k$ and $1 \leq j \leq 4$, the degree of v_j in H_i is 1. For, suppose without loss of generality that v_1 has degree at least 2 in H_1. Then, since G is 4-connected and S is independent, $H_1 - v_4$ has two independent paths from v_1 to v_2, v_3, respectively. These two paths and $P_2^2 \cup P_3^2$ form a cycle in $H_1 \cup H_2 - v_4$ containing $\{v_1, v_2, v_3\}$, contradicting (1).

For $1 \le j \le 4$, let v_j^i denote the unique neighbor of v_j in H_i. Let $H_i' := H_i/S$ be the graph obtained from H_i by identifying all vertices in S to a single vertex, denoted by u.

We claim that H_i' contains no subdivision of K_5. Otherwise, let K be a subdivision of K_5 in H_i'. If $u \notin V(K)$, then K is a subdivision of K_5 in G, a contradiction. If K uses precisely two edges at u, say uv_1^i and uv_2^i, then $K - u$, $v_1v_1^i$, $v_2v_2^i$, and $P_1^p \cup P_2^p$ (for some $p \in \{1, \dots, k\} - \{i\}$) form a subdivision of K_5 in G, a contradiction. Finally, if u is a branch vertex of K, then $K - u$, $v_jv_j^i$ and the paths P_j^p, $j = 1, 2, 3, 4$, for some $p \in \{1, \dots, k\} - \{i\}$, form a subdivision of K_5 in G, a contradiction.

Thus by the choice of G, each H_i' admits a 4-coloring, say c_i, and we may assume that $c_i(u)$, $i = 1, 2, 3, 4$, are the same. Let c be defined as follows: $c(x) = c_i(x)$ if $x \in V(H_i) - S$, and $c(x) = c_i(u)$ if $x \in S$. Clearly, c is a 4-coloring of G as S is independent, a contradiction. So we have (2).

(3) $G[S]$ has at most one edge.

For, suppose $G[S]$ has at least two edges. If $G[S]$ contains a path of length 2, say $v_1v_2v_3$, then P_1^1, P_3^1 and $v_1v_2v_3$ form a cycle in $H_1 - v_4$ containing $\{v_1, v_2, v_3\}$, which contradicts (1). So $G[S]$ contains exactly two edges which form a matching. Without loss of generality, let $v_1v_2, v_3v_4 \in E(G)$.

We claim that for any $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$, v_j has exactly one neighbor in C_i. For, suppose v_3 has at least two neighbors in C_2. Then by Menger's theorem, $H_2 - v_4$ has two independent paths from v_3 to v_1, v_2, respectively. Now these paths and v_1v_2 form a cycle in $H_2 - v_4$ containing $\{v_1, v_2, v_3\}$, a contradiction to (1).

Thus, for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$, let v_j^i denote the unique neighbor of v_j in C_i. By the choice of G, each C_i has a 4-coloring c_i. We may choose the colorings c_i (by renaming colors if necessary) such that $|\{c_i(v_1^i) : i = 1, 2, 3\}| = 1$ and $|\{c_i(v_3^i) : i = 1, 2, 3\}| \le 2$. It is now easy to see that the 4-colorings c_i $(i = 1, 2, 3)$ can be extended to a 4-coloring of G by greedily coloring v_4, v_3, v_2, v_1 in this order. This is a contradiction and completes the proof of (3).

(4) For each $l \in \{1, 2, 3\}$, $|\{x \in S : d_{H_l}(x) \ge 2\}| \le 2$.

Suppose that (4) is not true. Without loss of generality, assume v_1, v_2, v_3 all have at least two neighbors in H_1.

We claim $H_1 - v_4$ is 2-connected. For, suppose that $H_1 - v_4$ is not 2-connected. Let u be a cut vertex in $H_1 - v_4$. Then $u \notin S$ as C_1 is connected. Further, since G is 4-connected, each component of $H_1 - \{u, v_4\}$ contains one of $\{v_1, v_2, v_3\}$. So let D denote a component of $H_1 - \{u, v_4\}$ that contains exactly one vertex in $\{v_1, v_2, v_3\}$, say v_1. Then $V(D) = \{v_1\}$; as otherwise $G - \{u, v_1, v_4\}$ would not be connected. This shows that $v_1u, v_1v_4 \in E(G)$; in particular, v_4 also has at least two neighbors in H_1. So by repeating the above argument (for $H_1 - v_4$) on $H_1 - v_2$, we force $v_2v_j \in E(G)$ for some $j \in \{1, 3, 4\}$. But this contradicts (3).

By (1), $H_1 - v_4$ has no cycle containing $\{v_1, v_2, v_3\}$. Hence, (i) or (ii) or (iii) of Lemma 2.2 holds. Recall the notation in Lemma 2.2 and Figure 1. Since G is 4-connected, D_{v_i}, $i = 1, 2, 3$, are connected. Let $Z = G - (D_{v_1} \cup D_{v_2} \cup D_{v_3})$, and we choose $D_{v_1}, D_{v_2}, D_{v_3}$ to minimize Z. Then for (i) of Lemma 2.2 we have $Z = G[T]$. For (iii) of Lemma 2.2, we let Z_1, Z_2 be the components of Z, and for $i = 1, 2, 3$ we let $V(Z_1) \cap V(D_{v_i}) = \{a_i\}$ and $V(Z_2) \cap V(D_{v_i}) = \{b_i\}$. For (i) of Lemma 2.2, let $T = \{a_1, b_1\}$, $a_1 = a_2 = a_3$, and $b_1 = b_2 = b_3$. For (ii) of Lemma 2.2, let $T_{v_i} = \{a_i, b_i\}$ for $i = 1, 2, 3$, with $a_1 = a_2 = a_3$. It is straightforward to check that for $1 \leq i \neq j \leq 3$, $Z - \{b_1, b_2, b_3\}$ contains a path A_{ij} from a_i to a_j and $Z - \{a_1, a_2, a_3\}$ contains a path B_{ij} from b_i to b_j. Moreover, A_{ij} is disjoint from B_{kl} for all $1 \leq i \neq j \leq 3$ and $1 \leq k \neq l \leq 3$.

Note that v_4 cannot have neighbors in two of $D_{v_1}, D_{v_2}, D_{v_3}$. For, suppose v_4 has a neighbor in D_{v_1} as well as a neighbor in D_{v_2}. Then $G[V(D_{v_1}) \cup \{a_1, b_1, v_4\}]$ has a one of the following: a path Q_1 from v_4 to a_1 through v_1 and avoiding b_1, or a path R_1 from v_4 to b_1 through v_1 and avoiding a_1. (This can be done by applying Menger's theorem to the graph obtained from $G[V(D_{v_1}) \cup \{a_1, b_1\}]$ by identifying a_1 and b_1, which is 2-connected.) Similarly, $G[V(D_{v_2}) \cup \{a_2, b_2, v_4\}]$ has a path Q_2 from v_4 to a_2 through v_2 and avoiding b_2, or a path R_2 from v_4 to b_2 through v_2 and avoiding a_2. If Q_1, Q_2 (respectively, R_1, R_2) exist then Q_1, Q_2 (respectively, R_1, R_2) and A_{12} (respectively, B_{12}) form a cycle in $H_1 - v_3$ containing $\{v_1, v_2, v_4\}$, contradicting (1). Thus, without loss of generality we may assume that Q_1, R_2 exist, but neither Q_2 nor R_1 exists. Since R_1 does not exist, $G[V(D_{v_1}) \cup \{a_1, b_1, v_4\}]$ has a 2-separation (L_1, L_2) such that $a_1 \in V(L_1 \cap L_2)$, $v_1 \in V(L_1) - V(L_2)$ and $\{b_1, v_4\} \subseteq V(L_2)$. Since G is 4-connected, $|V(L_1)| = 3$ and a_1 has a neighbor in $L_2 - \{b_1, v_4\}$. Thus, since D_{v_1} is connected, $G[V(D_{v_1}) \cup \{a_1, v_4\}] - v_1$ has a path Q_1' from v_4 to a_1. We then have a cycle in $H - v_1$ containing $\{v_2, v_3, v_4\}$ using the following paths: $Q_1', R_2, A_{13}, B_{23}$, and a path in $G[V(D_3) \cup \{a_3, b_3\}]$ from a_3 to b_3 through v_3. This contradicts (1).

Thus we may assume by symmetry that neither D_{v_1} nor D_{v_2} contains a neighbor of v_4. Then, since G is 4-connected, $V(D_{v_i}) = \{v_i\}$ for $i = 1, 2$.

Case (4.1). Lemma 2.2(i) occurs.

If $H_1 - \{v_1, v_2\}$ contains a path from v_3 to v_4 and containing $\{a_1, b_1\}$, then such a path, $a_1 v_1 b_1 v_2 a_1$, $P_1^2 \cup P_2^2$, and $\cup_{j=1}^4 P_j^3$ form a subdivision of K_5 in G with branch set $\{a_1, b_1, v_1, v_2, x_3\}$, a contradiction. So such a path in $H_1 - \{v_1, v_2\}$ does not exist. Then $H_1' := H_1 - \{v_1, v_2\} + \{v, vv_3, vv_4\}$, where v is a new vertex, has no cycle containing $\{a_1, b_1, v\}$. Thus, (i) or (ii) or (iii) of Lemma 2.2 holds for H_1', a_1, b_1, v.

Suppose that Lemma 2.2(i) holds for H_1', a_1, b_1, v. Then $H_1 - \{v_1, v_2\}$ is the union of subgraphs $G_{a_1}, G_{b_1}, G_{v_3v_4}$, with pairwise intersection $\{s_1, s_2\}$, and has a 2-cut $\{s_1, s_2\}$ such that $a_1 \in V(G_{a_1}) - \{s_1, s_2\}$, $b_1 \in V(G_{b_1}) - \{s_1, s_2\}$, and $v_3, v_4 \in V(G_{v_3v_4})$. Since G is 4-connected, $V(G_{a_1}) = \{a_1, s_1, s_2\}$ and $V(G_{b_1}) = \{b_1, s_1, s_2\}$. Now $\{s_1, s_2, v_1, v_2\}$ is a 4-cut in G, and $G - \{s_1, s_2, v_1, v_2\}$ has three components, two of which are trivial, namely a_1 and b_1. By the choice of G, $G - a_1$ admits a 4-coloring, which can be extended to a 4-coloring of G by assigning the color of b_1 to a_1, a contradiction.

Now assume that Lemma 2.2(ii) holds for H_1', a_1, b_1, v. Then $H_1 - \{v_1, v_2\}$ is the union of subgraphs $G_1, G_2, G_{a_1}, G_{b_1}, G_{v_3v_4}$ and has 2-cuts $\{s_1, s_2\}$, $\{s_1, t_2\}$, $\{s_1, r_2\}$ such that s_2, t_2, r_2 are pairwise distinct, $a_1 \in V(G_{a_1}) - \{s_1, s_2\}$, $b_1 \in V(G_{b_2}) - \{s_1, t_2\}$, $v_3, v_4, r_1, r_2 \in V(G_{v_3v_4})$, $V(G_1) = \{s_1\}$, $s_1, s_2, t_2, r_2 \in V(G_2)$, $G_{a_1}, G_{b_1}, G_{v_3v_4}$ are pairwise disjoint, $V(G_2 \cap G_{a_1}) = \{s_1, s_2\}$, $V(G_2 \cap G_{b_1}) = \{s_1, t_2\}$, and $V(G_2 \cap G_{v_3v_4}) = \{s_1, r_2\}$. Since G is 4-connected, $V(G_{a_1}) = \{a_1, s_1, s_2\}$ and $V(G_{b_1}) = \{b_1, s_1, t_2\}$. Suppose s_1 and $\{r_2, s_2, t_2\}$ are contained in the same component of G_2. Let $v \in V(G_2) - \{s_1\}$. Since G is 4-connected, G_2 has four independent paths R_1, R_2, R_3, R_4 from v to s_1, r_2, s_2, t_2, respectively. Note that $G_{v_3v_4}$ has two disjoint paths from $\{v_3, v_4\}$ to $\{s_1, r_2\}$. Now these two paths, R_1, R_2, R_3, R_4,

$a_1v_2b_1s_1a_1$, s_2a_1, t_2b_1, $P_2^2 \cup P_3^2$, and $P_2^3 \cup P_4^3$ form a subdivision of K_5 in G with branch set $\{a_1, b_1, s_1, v, v_2\}$, a contradiction. Thus, s_1 and $\{r_2, s_2, t_2\}$ are not contained in the same component of G_2. Then $G_2 - a_1, G_{a_1}, G_{b_1}$ are triangles. We claim that $G_{v_3v_4}$ contains a path Q from s_1 to v_3 or v_4 and passing through r_2. For, otherwise, no path in $G_{v_3v_4}$ from s_1 to v_4 contains r_2. Then by Menger's theorem, $G_{v_3v_4}$ has a cut vertex w separating $\{s_1, v_4\}$ from r_2. Since G is 4-connected, the component of $G_{v_3v_4} - w$ containing r_2 must contain v_3 and has just two vertices (namely, r_2 and v_3), and $G_{v_3v_4}$ has a path from s_1 to v_3 and passing through r_2. So without loss of generality, assume Q is from s_1 to v_3. Now G has a subdivision of K_5 with branch set $\{a_1, s_1, s_2, t_2, r_2\}$ consisting of $G[\{s_1, s_2, t_2, r_2\}]$, $a_1s_1, a_1s_2, a_1v_2b_1t_2$, a_1v_1 and $P_1^2 \cup P_3^2$, and Q. This is a contradiction.

Thus, Lemma 2.2(iii) holds for H_1', a_1, b_1, v. Then $H_1 - \{v_1, v_2\}$ is the union of subgraphs $G_1, G_2, G_{a_1}, G_{b_1}, G_{v_3v_4}$ and has pairwise disjoint 2-cuts $\{s_1, s_2\}, \{t_1, t_2\}, \{r_1, r_2\}$ such that $a_1 \in V(G_{a_1}) - \{s_1, s_2\}$, $b_1 \in V(G_{b_2}) - \{t_1, t_2\}$, $v_3, v_4, r_1, r_2 \in V(G_{v_3v_4})$, $G_1 \cap G_2 = \emptyset$, the graphs $G_{a_1}, G_{b_1}, G_{v_3v_4}$ are pairwise disjoint, and for $i = 1, 2$, $V(G_i \cap G_{a_1}) = \{s_i\}$, $V(G_i \cap G_{b_1}) = \{t_i\}$, $V(G_i \cap G_{v_3v_4}) = \{r_i\}$. Since G is 4-connected, $V(G_{a_1}) = \{a_1, s_1, s_2\}$, $V(G_{b_1}) = \{b_1, t_1, t_2\}$, $V(G_i) = \{r_i, s_i, t_i\}$ for $i = 1, 2$, and the graphs $G_{a_1}, G_{b_1}, G_1, G_2$ are triangles. Then $\{v_1, v_2, r_1, r_2\}$ is a cut in G, and $G[\{a_1, b_1, s_1, s_2, t_1, t_2\}]$ is a component of $G - \{v_1, v_2, r_1, r_2\}$. By contracting $G[\{a_1, b_1, s_1, s_2, t_1, t_2\}]$ to a single vertex, say u, and noticing that $G[\{a_1, b_1, s_1, s_2, t_1, t_2\}]$ has four independent paths from a_1 to v_1, v_2, r_1, r_2, respectively, we obtain a graph from G that contains no subdivision of K_5 and hence admits a 4-coloring, say $c : V(G) \to \{1, 2, 3, 4\}$. Without loss of generality, assume $c(u) = 1$ and $c(r_1) = 2$. Let $c(a_1) = c(b_1) = 1$. If $c(r_2) = 2$ then let $c(s_1) = 3 = c(t_2)$ and $c(s_2) = 4 = c(t_1)$; so c becomes a 4-coloring of G, a contradiction. Hence, we may assume $c(r_2) = 3$. By letting $c(s_1) = 3$, $c(t_2) = 2$, $c(s_2) = 4 = c(t_1)$, we extend c to a 4-coloring of G, a contradiction.

Case (4.2). Lemma 2.2(ii) occurs.

First, assume a_1 and $\{b_1, b_2, b_3\}$ are in different components of Z. Then $a_ib_i \notin E(G)$ for $i = 1, 2, 3$. Thus, since G is 4-connected, $G[V(D_3) \cup \{a_1, b_3\}]$ has two independent paths from a_1 to v_3, b_3, receptively, and $G[(V(Z) - \{a_1\}) \cup \{v_4\}]$ has three independent paths from b_1 to b_2, b_3, v_4, respectively. Now the above five paths, $b_1v_1a_1, b_2v_2a_1$, $P_1^2 \cup P_2^2$, and P_j^3 (for $j = 1, 2, 3, 4$) form a subdivision of K_5 in G with branch set $\{a_1, b_1, v_1, v_2, x_3\}$, a contradiction.

Hence, Z is connected. Suppose v_4 has no neighbor in $Z - \{a_1, b_1, b_2, b_3\}$. Then, since G is 4-connected, for each $i \in \{1, 2\}$, Z has three independent paths Q_1^i, Q_2^i, Q_3^i from b_i to the three vertices in $\{a_1, b_1, b_2, b_3\} - \{b_i\}$, respectively. If $v_4b_1 \in E(G)$ then G has a subdivision of K_5 with branch set $\{a_1, b_1, v_1, v_2, x_3\}$ using two of $\{Q_1^1, Q_2^1, Q_3^1\}$ (which are from b_1 to a_1, b_2), $a_1v_1b_1, a_1v_2b_2$, $P_1^2 \cup P_2^2$, v_4b_1, a path in $G[V(D_3) \cup \{a_1\}]$ from v_3 to a_1, and P_j^3 $(1 \le j \le 4)$, a contradiction. Similarly, if $v_4b_2 \in E(G)$ we get a subdivision of K_5 in G with branch set $\{a_1, b_2, v_1, v_2, x_3\}$, a contradiction. Thus $v_4b_j \notin E(G)$ for $j = 1, 2$. Hence, v_4 has a neighbor in $V(D_3 - v_3) \cup \{a_1, b_3\}$. Since G is 4-connected, $G[V(D_3) \cup \{a_1, b_3, v_4\}]$ has two disjoint paths from $\{v_3, v_4\}$ to $\{a_1, b_3\}$. Now these two paths, Q_1^1, Q_2^1, Q_3^1, $a_1v_1b_1, a_1v_2b_2$, $P_1^2 \cup P_2^2$, and P_j^3 $(1 \le j \le 4)$ form a subdivision of K_5 in G with branch set $\{a_1, b_1, v_1, v_2, x_3\}$, a contradiction.

Thus, let $v \in V(Z) - \{a_1, b_1, b_2, b_3\}$ be a neighbor of v_4. We claim that there exist distinct $r, s \in \{1, 2, 3\}$ such that Z has three independent paths Q_1, Q_2, Q_3 from v to

a_1, b_r, b_s, respectively. If $Z - \{b_1, b_2, b_3\}$ does not contain a path from v to a_1 then, since G is 4-connected, Z has a path from a_1 to $\{b_1, b_2, b_3\}$ and three independent paths from v to b_1, b_2, b_3, respectively; so the claim holds. If $Z - \{b_1, b_2, b_3\}$ contains a path from v to a_1 then, since G is 4-connected, Z contains three independent paths from v to $\{a_1, b_1, b_2, b_3\}$; so the claim follows from Lemma 2.4.

If $\{r, s\} = \{1, 2\}$, then G has a subdivision of K_5 with branch set $\{a_1, v, v_1, v_2, x_3\}$ using Q_1, Q_2, Q_3, $a_1v_1b_1, a_1v_2b_2$, $P_1^2 \cup P_2^2$, a path in $G[V(D_3) \cup \{a_1\}]$ from v_3 to a_1, vv_4, and P_j^3 $(1 \leq j \leq 4)$, a contradiction. So assume by symmetry between D_1 and D_2 that $r = 1$ and $s = 3$. Then G has a subdivision of K_5 with branch set $\{a_1, v, v_1, v_3, x_3\}$ using Q_1, Q_2, Q_3, $a_1v_1b_1$, a path in $G[V(D_3) \cup \{a_1, b_3\}]$ from a_1 to b_3 through v_3, a_1v_2, $P_1^2 \cup P_3^2$, vv_4, and P_j^3 $(1 \leq j \leq 4)$, a contradiction.

Case (4.3). Lemma 2.2(iii) occurs.

Note that for each $i \in \{1, 2\}$, if v_4 has no neighbor in Z_i then, since G is 4-connected, Z_i is a triangle and $G[V(D_j) \cup \{a_j, b_j\}]$, $j = 1, 2$, are triangles.

Suppose v_4 has no neighbors in $Z_1 \cup Z_2$. So Z_1, Z_2 and $G[V(D_j) \cup \{a_j, b_j\}]$ (for $j = 1, 2$) are triangles. Note that $\{a_3, b_3, v_1, v_2\}$ is a 4-cut in G and $G[\{a_1, a_2, b_1, b_2\}]$ is a component of $G - \{a_3, b_3, v_1, v_2\}$. Contracting $G[\{a_1, a_2, b_1, b_2\}]$ to a single vertex u, we obtain from G a graph that contains no subdivision of K_5, and hence admits a 4-coloring. However this 4-coloring can be extended to a 4-coloring of G by assigning the color of u to a_1, b_2 and greedily coloring a_2, b_1. This is a contradiction.

Now assume that v_4 has a neighbor $v \in V(Z_1)$, but does not have a neighbor in Z_2. Then Z_2 and $G[V(D_j) \cup \{a_j, b_j\}]$ $(j = 1, 2)$ are triangles. Since G is 4-connected, $G[V(D_3) \cup \{a_3, b_3\}]$ has two independent paths from b_3 to a_3, v_3, respectively, and Z_1 has three independent paths from v to a_1, a_2, a_3, respectively. These five paths, Z_2, $v_ib_ia_i$ $(i = 1, 2)$, vv_4 and P_j^2 $(j = 1, 2, 3, 4)$ form a subdivision of K_5 in G with branch set $\{b_1, b_2, b_3, v, x_2\}$, a contradiction.

Similarly we get a contradiction if v_4 has a neighbor in Z_2, but does not have a neighbor in Z_1. So there exist $z_i \in V(Z_i)$ for $i = 1, 2$ such that $v_4z_i \in E(G)$.

First, assume $z_1 \notin \{a_1, a_2, a_3\}$ and $z_2 \notin \{b_1, b_2, b_3\}$. Since $H_1 - \{v_1, v_2, v_3, v_4\} = C_1$ is connected, there exists $r \in \{1, 2, 3\}$ such that $G[V(D_r) \cup \{a_r, b_r\}] - v_r$ contains a path Q_r from a_r to b_r. By the minimality of $Z_1 \cup Z_2$, Z_1 has two disjoint paths from $\{z_1, a_r\}$ to $\{a_1, a_2, a_3\} - \{a_r\}$, say R_1, R_2 from z_1, a_r to a_p, a_q, respectively, where $\{p, q\} = \{1, 2, 3\} - \{r\}$. If $G[V(Z_2) \cup \{v_4\}]$ has two disjoint paths from b_r to v_4 and from b_p to b_q, then these paths, R_1, R_2, Q_r, z_1v_4, and paths in $G[V(D_j) \cup \{a_j, b_j\}]$ (for $j \in \{p, q\}$) from a_j to b_j through v_j form a cycle in $H_1 - v_r$, contradicting (1). So such paths do not exist in $G[V(Z_2) \cup \{v_4\}]$. Since G is 4-connected, $G[V(Z_2) \cup \{v_4\}]$ is $(4, \{b_1, b_2, b_3, v_4\})$-connected; so by Lemma 2.3, $(G[V(Z_2) \cup \{v_4\}], b_r, b_p, v_4, b_q)$ is planar. Then, by the minimality of $Z_1 \cup Z_2$, $G[V(Z_2) \cup \{v_4\}]$ has disjoint paths from v_4 to b_q and from b_r to b_p. Now these paths, R_1, R_2, Q_r, and paths in $G[V(D_j) \cup \{a_j, b_j\}]$ (for $j \in \{p, q\}$) from a_j to b_j through v_j form a cycle in $H_1 - v_r$, contradicting (1).

Thus by symmetry we may assume that $z_1 \in \{a_1, a_2, a_3\}$ for every choice of z_1. Hence $V(Z_1) = \{a_1, a_2, a_3\}$ by the minimality of Z (and since G is 4-connected). If v_4 is adjacent to two of $\{a_1, a_2, a_3\}$, say a_k and a_l then we would get a cycle in $H - v_m$ (where $m \in \{1, 2, 3\} - \{k, l\}$) containing $\{v_k, v_l, v_4\}$ using $a_kv_4a_l$, paths in $G[D_j + \{a_j, b_j\}]$ (for $j = k, l$) between a_j and b_j through v_j, and B_{kl} (a path in Z_2 from b_k to b_l). This contradicts (1).

Thus, we may assume by symmetry that $v_4a_1 \notin E(G)$; so $v_4a_k \in E(G)$ for some $k \in \{2,3\}$ and $v_4a_{5-k} \notin E(G)$. Hence, since G is 4-connected, $G[V(D_1) \cup \{a_1, b_1\}]$ is a triangle.

We claim that z_2 must be b_k. For, suppose we may choose $z_2 \neq b_k$. Then by the minimality of $Z_1 \cup Z_2$, Z_2 contains disjoint paths R_1, R_2 from $\{b_1, b_{5-k}\}$ to z_2, b_k, respectively. By considering two cases $b_1 \in V(R_1)$ or $b_1 \in V(R_2)$, we see that R_1, R_2, $b_1a_1a_{5-k}, a_kv_4z_2$, and paths in $G[V(D_j) \cup \{a_j, b_j\}]$ (for $j = 2, 3$) from a_j to b_j through v_j form a cycle in $H_1 - v_1$ containing $\{v_2, v_3, v_4\}$, a contradiction.

We further claim that v_4 is not adjacent to D_{5-k}. For otherwise, since G is 4-connected, we may assume that $G[V(D_{5-k}) \cup \{a_{5-k}, b_{5-k}, v_4\}]$ has a path R from v_4 to a_{5-k} and through v_{5-k}, or from v_4 to b_{5-k} through v_{5-k}. Then we get a contradiction to (1) by finding a cycle in $H_1 - v_1$ containing $\{v_2, v_3, v_4\}$ using R, $a_{5-k}a_1b_1b_k \cup a_kv_4$ or $b_{5-k}b_1a_1a_k \cup b_kv_4$, and a path in $G[V(D_k) \cup \{a_k, b_k\}]$ from a_k to b_k through v_k.

Therefore, $G[V(D_{5-k} \cup \{a_{5-k}, b_{5-k}\}]$, Z_1, Z_2 are triangles. Now $\{a_k, b_k, v_1, v_{5-k}\}$ is a cut in G and $G[\{a_1, b_1, a_{5-k}, b_{5-k}\}]$ is a component of $G - \{a_k, b_k, v_1, v_{5-k}\}$. Contracting $G[\{a_1, b_1, a_{5-k}, b_{5-k}\}]$ to a single vertex u we obtain from G a graph containing no subdivision of K_5, and hence admitting a 4-coloring, say c. Now c can be extended to a 4-coloring of G by coloring a_1 and b_{5-k} with $c(u)$, and then greedily coloring b_1 and a_{5-k}. This is a contradiction and completes the proof of (4).

By (2) and (3) we may assume that v_1 and v_2 each have at least two neighbors in H_1, and v_1v_2 is the only possible edge in $G[S]$. Then by (4), v_3 and v_4 each have degree 1 in H_1. Let H be obtained from $H_2 \cup H_3$ by adding v_1v_2 (if not already present) and identifying v_2 and v_3. Let v denote the identification of v_2 and v_3.

(5) H contains no subdivision of K_5.

Suppose to the contrary that K is a subdivision of K_5 in H. Then $v \in V(K)$ as otherwise K would be a subdivision of K_5 in G.

Suppose v is not a branch vertex of K. Let vu_1, vu_2 denote the edges of K incident with v. If there exists $p \in \{2,3\}$ such that $v_pu_1, v_pu_2 \in E(G)$ then $K - v$ and $u_1v_pu_2$ form a subdivision of K_5 in G, a contradiction. So without loss of generality assume that $v_2u_1, v_3u_2 \in E(G)$. Then $K - v$, $P_2^1 \cup P_3^1$, v_2u_1 and v_3u_2 form a subdivision of K_5 in G, a contradiction.

Thus v must be a branch vertex of K. Let vu_i, $i = 1, 2, 3, 4$, be the edges of K incident with v. If there exists $p \in \{2, 3\}$ such that $v_pu_i \in E(G)$ for $i = 1, 2, 3, 4$, then $K - v$ and v_pu_i $(i = 1, 2, 3, 4)$ form a subdivision of K_5 in G, a contradiction.

Suppose there exists $p \in \{2, 3\}$ such that v_p is adjacent to three of $\{u_i : i = 1, 2, 3, 4\}$, say u_1, u_2, u_3, and $v_{5-p}u_4 \in E(G)$. Then $K - v$, $P_2^1 \cup P_3^1$, v_pu_i $(i = 1, 2, 3)$ and $v_{5-p}u_3$ form a subdivision of K_5 in G, a contradiction.

Thus, we may assume that v_2 is adjacent to u_1 and u_2, and v_3 is adjacent to u_3 and u_4. Note that all branch vertices of K must be contained in $V(C_l) \cup \{v, v_1, v_4\}$ for some $l \in \{2, 3\}$, say $l = 3$; for, otherwise both C_2 and C_3 would contain a branch vertex of K, however H cannot have four independent paths between them (but K does), a contradiction. Moreover, C_3 contains at least two branch vertices of K, and $|\{u_1, u_2, u_3, u_4\} \cap V(C_3)| \geq 2$. Let Q_i denote the path in K between v and another branch vertex and containing vu_i (but no other branch vertex is internal to Q_i).

Suppose $|\{u_1, u_2, u_3, u_4\} \cap V(C_3)| = 3$. First, assume $u_1 \in V(C_2)$ and $u_2, u_3, u_4 \in V(C_3)$. Note that Q_1 must pass through $\{v_1, v_4\}$ before entering $V(C_3) \cup \{v_1, v_4\}$. We may assume that Q_1 enters C_3 through v_1. We can now produce a subdivision of K_5 in G from K by replacing vQ_1v_1 with $P_1^1 \cup P_3^1$, replacing vu_2 with $P_2^2 \cup P_3^2$ and v_2u_2, and replacing vu_3 with v_3u_3, and replacing vu_4 with v_3u_4. This is a contradiction. Similarly, we derive a contradiction for other cases when $|\{u_1, u_2, u_3, u_4\} \cap V(C_3)| = 3$.

Now suppose $|\{u_1, u_2, u_3, u_4\} \cap V(C_3)| = 2$. First, assume $u_1, u_2 \in V(C_2)$ and $u_3, u_4 \in V(C_3)$. Note that Q_1, Q_2 must go through $\{v_1, v_4\}$ before entering $V(C_3) \cup \{v_1, v_4\}$. So we may assume by symmetry that $v_1 \in V(Q_1)$ and $v_4 \in V(Q_2)$. We get a subdivision of K_5 in G from K by replacing vQ_1v_1 with $P_1^1 \cup P_3^1$, replacing vQ_2v_4 with $P_3^2 \cup P_4^2$, replacing vu_3 with v_3u_3, and replacing vu_4 with v_3u_4. This is a contradiction. Similarly, we derive a contradiction if $u_1, u_2 \in V(C_3)$ and $u_3, u_4 \in V(C_2)$.

Therefore, we may assume without loss of generality that $u_1, u_3 \in V(C_2)$ and $u_2, u_4 \in V(C_3)$. Now Q_1, Q_3 must pass through $\{v_1, v_4\}$ before entering $V(C_3) \cup \{v_1, v_4\}$. So we may assume $v_1 \in V(Q_1)$ and $v_4 \in V(Q_3)$. We obtain from K a subdivision of K_5 in G by replacing vQ_1v_1 and vQ_3v_4 with P_j^1 $(1 \le j \le 4)$, replacing vu_2 with v_2u_2, and replacing vu_4 with v_3u_4. This is a contradiction and completes the proof of (5).

By (5), we see that H is 4-colorable. Let $c : V(H) \to \{1, 2, 3, 4\}$ be a 4-coloring of H. Without loss of generality we may assume $c(v_1) = 1$ and $c(v) = 2$. We may also view c as a 4-coloring of $H_2 \cup H_3$, with $c(v_2) = c(v_3) = 2$. We now extend this coloring to G. For $i = 3, 4$, let t_i denote the neighbor of v_i in C_1.

Case 1. $c(v_4) \in \{3, 4\}$.

Without loss of generality, let $c(v_4) = 3$. Let $H_1' := (H_1 - \{v_3, v_4\}) + \{v_1v_2, v_2t_3\}$. If H_1' contains no subdivision of K_5 then H_1' admits a 4-coloring $c' : V(H_1') \to \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $c'(v_1) = 1$ and $c'(v_2) = 2$. Then $c'(t_3) \neq 2$. We may also assume $c'(t_4) \neq 3$, otherwise we can exchange the colors 3 and 4 for c'. It is easy to see that c and c' combined gives a 4-coloring of G, a contradiction.

So H_1' contains a subdivision of K_5, say K. Let K' be obtained from K as follows: if $v_1v_2 \in E(K)$ we replace v_1v_2 with $P_1^2 \cup P_2^2$, and if $v_2t_3 \in E(K)$ we replace v_2t_3 with $P_2^3 \cup P_3^3$ and v_3t_3. Then K' is a subdivision of K_5 in G, a contradiction.

Case 2. $c(v_4) = 2$.

Let $H_1' := (H_1 - \{v_3, v_4\}) + \{v_1v_2, v_2t_3, v_2t_4\}$. If H_1' has no subdivision of K_5 then H_1' admits a 4-coloring $c' : V(H_1') \to \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $c'(v_1) = 1$ and $c'(v_2) = 2$. Then $c'(t_3) \neq 2$ and $c'(t_4) \neq 2$. It is easy to see that c and c' combined gives a 4-coloring of G, a contradiction.

So H_1' must contain a subdivision of K_5, say K. Then $v_2 \in V(K)$ as otherwise K would be a subdivision of K_5 in G. If $v_2v_1, v_2t_3, v_2t_4 \in E(K)$ then replacing them with P_j^2 $(1 \le j \le 4)$, v_3t_3, and v_4t_4, we produce a subdivision of K_5 in G, a contradiction. Now suppose exactly one of $\{v_2v_1, v_2t_3, v_2t_4\}$ is in K. We get a contradiction by finding a subdivision of K_5 in G from K as follows: if $v_2v_1 \in E(K)$ we replace it with $P_1^2 \cup P_2^2$, if $v_2t_3 \in E(K)$ we replace it with $P_2^2 \cup P_3^2$ and v_3t_3, and if $v_2t_4 \in E(K)$ we replace it with $P_2^2 \cup P_4^2$ and v_4t_4. So exactly two of $\{v_2v_1, v_2t_3, v_2t_4\}$ are in K. Let K' be obtained from K as follows. If $v_2v_1, v_2t_3 \in E(K)$ we replace v_2v_1 with $P_1^2 \cup P_2^2$, and replace v_2t_3 with $P_2^3 \cup P_3^3$ and v_3t_3. If $v_2v_1, v_2t_4 \in E(K)$ we replace v_2v_1 with $P_1^2 \cup P_2^2$, and replace v_2t_4 with $P_2^3 \cup P_4^3$

and v_4t_4. If $v_2t_3, v_2t_4 \in E(K)$ we replace v_2t_3 with $P_2^2 \cup P_3^2$ and v_3t_3, and replace v_2t_4 with $P_2^3 \cup P_4^3$ and v_4t_4. In each case we see that K' is a subdivision of K_5 in G, a contradiction.

Case 3. $c(v_4) = 1$.

Let $H_1' := (H_1 - \{v_3, v_4\}) + \{v_1v_2, v_2t_3, v_1t_4\}$. If H_1' has no subdivision of K_5 then H_1' admits a 4-coloring, say $c' : V(H_1') \to \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $c'(v_1) = 1$ and $c'(v_2) = 2$. Then $c'(t_3) \neq 2$ and $c'(t_4) \neq 1$. It is easy to see that c and c' combined gives a 4-coloring of G, a contradiction.

So H_1' contains a subdivision of K_5, and we denote it by K. If H_2 contains disjoint paths Q_1, Q_2 from v_1, v_2 to v_4, v_3, respectively, then we obtain a contradiction by constructing a subdivision of K_5 in G from K as follows: If $v_1t_4 \in E(K)$ we replace v_1t_4 with Q_1 and v_4t_4, if $v_2t_3 \in E(K)$ we replace v_2t_3 with Q_2 and v_3t_3, and if $v_1v_2 \in E(K)$ we replace it with $P_1^3 \cup P_2^3$. So Q_1, Q_2 do not exist in H_2; hence by Lemma 2.3, $(H_2, v_1, v_2, v_4, v_3)$ is planar. Similarly, $(H_3, v_1, v_2, v_4, v_3)$ is planar. Thus, $(H_2 \cup H_3) + v_1v_2$ is a planar graph, and identifying v_2, v_3, v_4 results in a planar graph H'. So H' contains no subdivision of K_5, and hence admits a 4-coloring. Now this 4-coloring of H' gives a 4-coloring c^* of $H_2 \cup H_3$ for which we may assume $c^*(v_1) = 1$, $c^*(v_2) = c^*(v_3) = c^*(v_4) = 2$. With c^* replacing c, we get back to Case 2. This completes the proof of Theorem 1.1. ∎

Acknowledgment. We thank an anonymous referee for discovering a missing case in the original manuscript, and for an observation that led to a more direct argument in Case (4.1) in Section 3.

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