On judicious bipartitions of graphs

Jie Ma*
Xingxing Yu†

Abstract

For a positive integer m, let $f(m)$ be the maximum value t such that any graph with m edges has a bipartite subgraph of size at least t, and let $g(m)$ be the minimum value s such that for any graph G with m edges there exists a bipartition $V(G) = V_1 \cup V_2$ such that G has at most s edges with both incident vertices in V_i. Alon proved that the limsup of $f(m) - (m/2 + \sqrt{m/8})$ tends to infinity as m tends to infinity, establishing a conjecture of Erdős. Bollobás and Scott proposed the following judicious version of Erdős' conjecture: the limsup of $m/4 + \sqrt{m/32} - g(m)$ tends to infinity as m tends to infinity. In this paper, we confirm this conjecture. We also generalize Alon's result to k-partitions, which should be useful for generalizing the above Bollobás-Scott conjecture to k-partitions.

*Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email:jiema@math.ucla.edu
†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160. Partially supported by NSA

1 Introduction

Let G be a graph and $e \in E(G)$. We will use $V(e)$ to denote the set of the vertices of G incident with e. Let $S, T \subseteq V(G)$. We write $e_G(S) := |\{e \in E(G) : V(e) \subseteq S\}|$, $(S,T)_G := \{e \in E(G) : V(e) \cap S \neq \emptyset \neq V(e) \cap T\}$, and $e_G(S,T) := |(S,T)_G|$. If $S = \{v\}$, then we simply write $e_G(v,T)$ for $e_G(\{v\},T)$. For any integer $k \geq 2$ and any k-partition $V_1, V_2, ..., V_k$ of $V(G)$, let $e_G(V_1, V_2, ..., V_k) = \sum_{1\leq i<j\leq k} e(V_i, V_j)$. When understood, the reference to G in the subscript will be dropped.

Again let G be a graph. We write $e(G) := |E(G)|$. For any integer $k \geq 2$, let $f_k(G)$ denote the maximum number of edges in a k-partite subgraph of G. For any integers $k \geq 2$ and $m \geq 1$, let $f_k(m) := \min\{f_k(G) : e(G) = m\}$. Let $f(G) := f_2(G)$ and $f(m) := f_2(m)$.

The problem for deciding $f(G)$ is NP-hard, and there has been extensive work on approximating $f(G)$, see [3, 13–15, 19]. On the other hand, it is easy to prove that any graph with m edges has a partition V_1, V_2 with $e(V_1, V_2) \geq m/2$. Edwards [10, 11] improved this lower bound by showing

$$f(m) \geq \frac{m}{2} + \frac{1}{4}\left(\sqrt{2m + 1/4} - \frac{1}{2}\right) = m/2 + \sqrt{m/8} + O(1).$$

This is best possible, as K_{2n+1} are extremal graphs. Erdős [12] conjectured that the limsup of

$$f(m) - (m/2 + \sqrt{m/8})$$

tends to infinity as m tends to infinity. Alon [1] confirmed this conjecture with the following

Theorem 1.1 (*Alon*) *There exist absolute constants $c > 0$ and $M > 0$ such that for every even integer $n > M$, if $m = n^2/2$ then*

$$f(m) \geq m/2 + \sqrt{m/8} + cm^{1/4}.$$

Bollobás and Scott [4] considered problems in which one needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *judicious partitioning problems*, and we refer the reader to [6–8, 16] for more problems and references.

Let G be a graph and k a positive integer; we use $g_k(G)$ to denote the minimal number t such that there exists a k-partition $V(G) = V_1 \cup \ldots \cup V_k$ satisfying $e(V_i) \leq t$ for $i = 1, ..., k$. For any positive integers k and m, let $g_k(m) := \max\{g_k(G) : e(G) = m\}$. Let $g(G) := g_2(G)$ and $g(m) := g_2(m)$.

The problem of determining $g(G)$ is known as the *Bottleneck Graph Bipartition Problem* and is shown to be NP-hard, see [17]. On the other hand, Bollobás and Scott [5] showed that for any positive integer m

$$g(m) \leq \frac{m}{4} + \frac{1}{8}(\sqrt{2m + 1/4} - 1/2) = m/4 + \sqrt{m/32} + O(1).$$

This bound is sharp, as K_{2n+1} are extremal graphs. Bollobás and Scott [6] (also see [16]) proposed the following judicious version of Erdős' conjecture.

Conjecture 1.2 (*Bollobás and Scott*) *The limsup of*

$$m/4 + \sqrt{m/32} - g(m)$$

tends to infinity as m tends to infinity.

The main goal of this paper is to prove the following result which confirms Conjecture 1.2. Theorem 1.1 plays an important role in our proof.

Theorem 1.3 *There exist absolute constants $d > 0$ and $N > 0$ such that for every even integer $n > N$, if $m = n^2/2$ then*

$$g(m) \le m/4 + \sqrt{m/32} - dm^{1/4}.$$

Bollobás and Scott [7] (also see [6]) extended Edwards' bound to k-partitions of graphs and proved that the vertex set of any graph with m edges can be partitioned into $V_1, \dots, V_k$ such that

$$e(V_1, \dots, V_k) \ge \frac{k-1}{k}m + \frac{k-1}{2k}(\sqrt{2m+1/4} - 1/2) + O(k),$$

with equality when G is the complete graph of order $kn+1$. Bollobás and Scott [5] also showed that the vertex set of any graph G with m edges can be partitioned into $V_1, \dots, V_k$ such that for $i \in \{1, 2, \dots, k\}$,

$$e(V_i) \le \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2).$$

Recently, Xu and Yu [18] showed that a partition $V_1, \dots, V_k$ can be found that satisfies both inequalities simultaneously, generalizing an earlier bipartition result of Bollobás and Scott [5].

It is natural to ask whether Theorems 1.1 and 1.3 can be extended to k-partitions. Here we present the following generalization of Theorem 1.1, which we think should be useful for extending Theorem 1.3 to k-partitions.

Theorem 1.4 *For any integer $k \ge 2$, there exist positive constants $c(k) = O(1/\sqrt{k})$ and $N(k) = O(k^3)$ such that for every even integer $n > N(k)$, if $m = n^2/2$ then*

$$f_k(m) \ge \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + c(k)m^{1/4}.$$

We organize this paper as follows. In Section 2, we prove Theorem 1.3. In Section 3, we prove Theorem 1.4 by refining Alon's original proof of Theorem 1.1. In Section 4, we offer some concluding remarks.

2 Bipartitions

In this section, we prove Theorem 1.3. Our proof is divided into several cases according to values of $f(G) - e(G)/2$. Since we will be using maximum bipartite subgraphs to find good judicious partitions, we need a result from [2] which establishes a connection between $f(G)$ and $g(G)$; a similar connection between $f_k(G)$ and $g_k(G)$ can be found in Bollobás and Scott [9].

Lemma 2.1 (*Alon, Bollobás, Krivelevich and Sudakov*) *Let G be a graph with m edges and $f(G) = m/2+\delta$. If $\delta \geq m/30$ then there exists an absolute constant D such that, when $m \geq D$ there exists a bipartition $V(G) = V_1 \cup V_2$ satisfying*

$$\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} - \frac{m}{100}.$$

If $\delta \leq m/30$, then there exists a partition $V(G) = V_1 \cup V_2$ such that

$$\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}.$$

Lemma 2.1 implies Theorem 1.3 for graphs G with $e(G) = m$ and $f(G) - m/2 = \delta \geq m/30$. The following easy consequence of Lemma 2.1 proves Theorem 1.3 for graphs G with $e(G) = m$ and $f(G) - m/2 \geq m/10^4$.

Lemma 2.2 *There exists an absolute constant $M_1 > 0$ such that the following holds. If G is a graph with m edges and $f(G) = m/2 + \delta$, $m \geq M_1$, and $\delta \geq m/10^4$, then there exists a bipartition $V(G) = V_1 \cup V_2$ such that for $i = 1, 2$,*

$$e(V_i) \leq \frac{m}{4} - \frac{m}{4 \times 10^4}.$$

Proof. Let G be a graph with m edges and $f(G) = m/2 + \delta$, and assume $\delta \geq m/10^4$. By Lemma 2.1, we may assume that $m/10^4 \leq \delta \leq m/30$ (so we will require $M_1 \geq D$) and that there exists a partition $V(G) = V_1 \cup V_2$ such that for $i = 1, 2$,

$$e(V_i) \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}.$$

Let $h(\delta) := -\delta/2 + 10\delta^2/m$. Differentiating with respect to δ, we have $h'(\delta) = -1/2 + 20\delta/m$. So $h'(\delta) \geq 0$ when $\delta \geq m/40$, and $h'(\delta) \leq 0$ when $\delta \leq m/40$. Thus, since $m/10^4 \leq \delta \leq m/30$,

$$h(\delta) \leq \max\{h(m/10^4), h(m/30)\} = \max\left\{-\frac{1}{2}\frac{m}{10^4} + \frac{m}{10^7}, -\frac{m}{60} + \frac{m}{90}\right\} = -\frac{1}{2}\frac{m}{10^4} + \frac{m}{10^7}.$$

Hence, for $i = 1, 2$,

$$e(V_i) \leq \frac{m}{4} + 3\sqrt{m} - \frac{1}{2}\frac{m}{10^4} + \frac{m}{10^7}.$$

Clearly, there exists constant $M_1 \geq D > 0$ such that when $m \geq M_1$,

$$3\sqrt{m} - \frac{1}{2}\frac{m}{10^4} + \frac{m}{10^7} \leq -\frac{m}{4 \times 10^4}.$$

Hence $e(V_i) \leq m/4 - m/(4 \times 10^4)$. ∎

In view of Lemma 2.2, it suffices to prove Theorem 1.3 for graphs G with $f(G) - m/2 \leq m/10^4$. Hence we need to find special vertices that we could use to modify existing bipartitions. This will be done in the next lemma and Lemma 2.5, using an approach similar to ones in [2, 5, 9].

Lemma 2.3 *Let G be a graph with m edges, and assume $f(G) = m/2 + \delta$, where $\delta \le m/10^4$. Suppose $V(G) = V_1 \cup V_2$ is a partition such that $e(v, V_1) \le e(v, V_2)$ for every $v \in V_1$, and $e(V_1) \ge m/4 - \delta/2$. Then there exists $v \in V_1$ such that*

$$e(v, V_1) \le \sqrt{m/2} + 3\sqrt{\delta} \text{ and } e(v, V_2) \le (1 + 20\sqrt{\delta/m})e(v, V_1).$$

Proof. Let

$$T = \{v \in V_1 : e(v, V_1) > \sqrt{m/2} + 3\sqrt{\delta}\}$$

and

$$S = \{v \in V_1 : e(v, V_2) > \left(1 + 20\sqrt{\delta/m}\right) e(v, V_1)\}.$$

We will show that $\sum_{v \in V_1} e(v, V_1) > \sum_{v \in S \cup T} e(v, V_1)$ from which the existence of the desired vertex v follows. To this end, we bound $\sum_{v \in T} e(v, V_1)$ and $\sum_{v \in S} e(v, V_1)$.

Since $e(v, V_1) \le e(v, V_2)$ for all $v \in V_1$, we have

$$e(V_1) = \frac{1}{2}\sum_{v \in V_1} e(v, V_1) \le \frac{1}{2}\sum_{v \in V_1} e(v, V_2) = \frac{1}{2}e(V_1, V_2) \le \frac{f(G)}{2} = \frac{m}{4} + \frac{\delta}{2}.$$

On the other hand,

$$2e(V_1) = \sum_{v \in V_1} e(v, V_1) \ge \sum_{v \in T} e(v, V_1) > \left(\sqrt{m/2} + 3\sqrt{\delta}\right)|T|.$$

Therefore

$$|T| < \frac{2e(V_1)}{\sqrt{m/2} + 3\sqrt{\delta}} \le \frac{m/2 + \delta}{\sqrt{m/2} + 3\sqrt{\delta}} < \sqrt{m/2} - \frac{3}{2}\sqrt{\delta},$$

where the final inequality holds because $\delta \le m/10^4$. Hence

$$\sum_{v \in T} e(v, V_1) \le e(V_1) + e(T) < e(V_1) + \frac{1}{2}|T|^2 < e(V_1) + \frac{1}{2}\left(\sqrt{m/2} - \frac{3}{2}\sqrt{\delta}\right)^2.$$

Since $\delta \le m/10^4$,

$$\sum_{v \in T} e(v, V_1) < e(V_1) + \frac{m}{4} - \frac{3}{4}\sqrt{m\delta/2}.$$

Note that $e(V_1, V_2) = \sum_{v \in S} e(v, V_2) + \sum_{v \in V_1 - S} e(v, V_2)$; so

$$e(V_1, V_2) \ge \left(1 + 20\sqrt{\delta/m}\right)\sum_{v \in S} e(v, V_1) + \sum_{v \in V_1 - S} e(v, V_1) = 2e(V_1) + 20\sqrt{\delta/m}\sum_{v \in S} e(v, V_1).$$

Therefore, since $e(V_1) \ge m/4 - \delta/2$ and $e(V_1, V_2) \le f(G) = m/2 + \delta$,

$$\sum_{v \in S} e(v, V_1) < \frac{1}{20}\sqrt{m/\delta}(e(V_1, V_2) - 2e(V_1)) \le \frac{1}{20}\sqrt{m/\delta}\left(\frac{m}{2} + \delta - 2\left(\frac{m}{4} - \frac{\delta}{2}\right)\right) = \frac{1}{10}\sqrt{m\delta}.$$

Combining the above bounds on $\sum_{v \in T} e(v, V_1)$ and $\sum_{v \in S} e(v, V_1)$, we have

$$\sum_{v \in S \cup T} e(v, V_1) \le \sum_{v \in S} e(v, V_1) + \sum_{v \in T} e(v, V_1) < e(V_1) + \frac{m}{4} - \frac{3}{8}\sqrt{m\delta/2}.$$

Since $e(V_1) \geq m/4 - \delta/2$, we have

$$\sum_{v \in V_1} e(v, V_1) = 2e(V_1) \geq e(V_1) + \frac{m}{4} - \frac{\delta}{2} > e(V_1) + \frac{m}{4} - \frac{3}{8}\sqrt{m\delta/2} > \sum_{v \in S \cup T} e(v, V_1),$$

where the second equality holds because $\delta \leq m/10^4$. So $V_1 - (S \cup T) \neq \emptyset$ and the desired vertex exists. ∎

The following result says that there exists constant $C > 0$ such that Theorem 1.3 holds for graphs G with m edges and $\sqrt{m/8} + Cm^{1/4} \leq f(G) - m/2 \leq m/10^4$.

Lemma 2.4 *Let c be the constant in Theorem 1.1, let $d = c/4$, and let $C := 2(70 + d)$. There exists an absolute constant $M_2 > 0$ such that the following holds. If G is a graph with m edges, $f(G) = m/2 + \delta$, $m \geq M_2$, and $\sqrt{m/8} + Cm^{1/4} \leq \delta \leq m/10^4$, then there exists a partition $V(G) = V_1 \cup V_2$ such that for $i = 1, 2$,*

$$e(V_i) \leq m/4 + \sqrt{m/32} - dm^{1/4}.$$

Proof. Let G be a graph with m edges and $f(G) = m/2+\delta$, and assume that $\sqrt{m/8}+Cm^{1/4} \leq f(G) - m/2 \leq m/10^4$. Let $V(G) = U_1 \cup U_2$ be a partition such that

$$e(U_1, U_2) = f(G) = \frac{m}{2} + \delta.$$

Without loss of generality we may assume $e(U_1) \geq e(U_2)$. Note that $e(v, U_1) \leq e(v, U_2)$ for every $v \in U_1$; otherwise, $e(U_1 - \{v\}, U_2 \cup \{v\}) > e(U_1, U_2) = f(G)$, a contradiction. Hence

$$e(U_1) = \frac{1}{2}\sum_{v \in U_1} e(v, U_1) \leq \frac{1}{2}\sum_{v \in U_1} e(v, U_2) = \frac{1}{2}e(U_1, U_2) = \frac{m}{4} + \frac{\delta}{2}.$$

We now define a process to move vertices from U_1 to U_2, using Lemma 2.3, such that in the end we get the desired partition.

- Initially, we set $V_1^0 := U_1, V_2^0 := U_2$. Let $V_1^i \cup V_2^i$ denote the partition of $V(G)$ after the ith iteration.
- If $e(V_1^i) \leq m/4 - \delta/2 + (\sqrt{m/2} + 3\sqrt{\delta})/2$, set $V_1 := V_1^i$ and $V_2 := V_2^i$, and stop.
- If $e(V_1^i) > m/4 - \delta/2 + (\sqrt{m/2} + 3\sqrt{\delta})/2$ then by Lemma 2.3, there exists $u_i \in V_1^i$ such that

  $$e(u_i, V_1^i) \leq \sqrt{m/2} + 3\sqrt{\delta} \quad \text{and} \quad e(u_i, V_2^i) \leq \left(1 + 20\sqrt{\delta/m}\right) e(u_i, V_1^i).$$

  Set $V_1^{i+1} := V_1^i - \{u_i\}, V_2^{i+1} := V_2^i \cup \{u_i\}$, and repeat the above steps for V_1^{i+1}, V_2^{i+1}.

Note that, after i iterations (for each i), we always have $e(v, V_1^i) \leq e(v, V_2^i)$ for every $v \in V_1^i$; so Lemma 2.3 may be applied to V_1^i, V_2^i. Let $V_1 = V_1^k, V_2 = V_2^k$ denote the final partition, after k iterations. Then

$$e(V_1) \leq \frac{m}{4} - \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta}.$$

Moreover, since V_1 is obtained from V_1^{k-1} by moving u_{k-1} to V_2^{k-1}, we have

$$e(V_1) > \frac{m}{4} - \frac{\delta}{2} - \frac{1}{2}(\sqrt{m/2} + 3\sqrt{\delta}).$$

Since $e(V_1) = e(U_1) - \sum_{i=0}^{k-1} e(u_i, V_1^i)$, we have

$$\sum_{i=0}^{k-1} e(u_i, V_1^i) = e(U_1) - e(V_1) < e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}(\sqrt{m/2} + 3\sqrt{\delta}).$$

Hence, since $e(U_2) = m - e(U_1, U_2) - e(U_1) = m/2 - \delta - e(U_1)$ and $e(u_i, V_2^i) < (1 + 20\sqrt{\delta/m})e(u_i, V_1^i)$, we have

$$\begin{aligned}
e(V_2) &= e(U_2) + \sum_{i=0}^{k-1} e(u_i, V_2^i) \\
&\le \frac{m}{2} - \delta - e(U_1) + (1 + 20\sqrt{\delta/m}) \sum_{i=0}^{k-1} e(u_i, V_1^i) \\
&\le \frac{m}{2} - \delta - e(U_1) + (1 + 20\sqrt{\delta/m}) \left(e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \\
&= \frac{m}{4} - \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} + 20\sqrt{\delta/m} \left(e(U_1) - \frac{m}{4} + \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \\
&\le \frac{m}{4} - \frac{\delta}{2} + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} + 20\sqrt{\delta/m} \left(\delta + \frac{1}{2}\sqrt{m/2} + \frac{3}{2}\sqrt{\delta} \right) \quad (\text{since } e(U_1) \le m/4 + \delta/2) \\
&= \frac{m}{4} - \frac{\delta}{2} + \sqrt{m/8} + (3/2 + 5\sqrt{2})\sqrt{\delta} + 20\frac{\delta^{3/2}}{\sqrt{m}} + 30\frac{\delta}{\sqrt{m}}.
\end{aligned}$$

Therefore,

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} + \sqrt{m/8} - \frac{\delta}{2} + (3/2 + 5\sqrt{2})\sqrt{\delta} + 20\frac{\delta^{3/2}}{\sqrt{m}} + 30\frac{\delta}{\sqrt{m}}.$$

Let

$$h(\delta) := -\frac{\delta}{2} + (3/2 + 5\sqrt{2})\sqrt{\delta} + 20\frac{\delta^{3/2}}{\sqrt{m}} + 30\frac{\delta}{\sqrt{m}}.$$

Differentiating with respect to δ, we have

$$h'(\delta) = -\frac{1}{2} + (3/2 + 5\sqrt{2})\frac{1}{2\sqrt{\delta}} + \frac{30}{\sqrt{m}}\sqrt{\delta} + \frac{30}{\sqrt{m}}.$$

Since $C = 2(70 + d)$ is an absolute constant and because $\sqrt{m/8} + Cm^{1/4} \le \delta \le m/10^4$, it is easy to see that there exists an absolute constant $M_2 > 0$ such that if $m \ge M_2$ then $h'(\delta) < 0$ and $\sqrt{m/8} + Cm^{1/4} < \sqrt{m}$.

Thus, $h(\delta)$ is a decreasing function when $m \geq M_2$. Hence, if $m \geq M_2$ then

$$
\begin{aligned}
&\max\{e(V_1), e(V_2)\} \\
\leq &\frac{m}{4} + \sqrt{m/8} + h(\delta) \\
\leq &\frac{m}{4} + \sqrt{m/8} + h(\sqrt{m/8} + Cm^{1/4}) \\
= &\frac{m}{4} + \sqrt{m/8} - \frac{\sqrt{m/8} + Cm^{1/4}}{2} + (3/2 + 5\sqrt{2})\sqrt{\sqrt{m/8} + Cm^{1/4}} \\
&\quad + \frac{20}{\sqrt{m}}\left(\sqrt{m/8} + Cm^{1/4}\right)^{3/2} + 30\frac{\sqrt{m/8} + Cm^{1/4}}{\sqrt{m}} \\
< &\frac{m}{4} + \sqrt{m/32} - \frac{C}{2}m^{1/4} + 20m^{1/4} + 20m^{1/4} + 30m^{1/4} \quad (\text{since } \sqrt{m/8} + Cm^{1/4} < \sqrt{m}) \\
= &\frac{m}{4} + \sqrt{m/32} - dm^{1/4} \quad (\text{since } C = 2(70 + d)),
\end{aligned}
$$

completing the proof of this lemma. ∎

The next result is similar to Lemma 2.3 which is needed to prove Theorem 1.3 for graphs G with m edges and $f(G) - m/2 \leq \sqrt{m/8} + Cm^{1/4}$.

Lemma 2.5 *Let c be the constant in Theorem 1.1 and d, C be the constants in Lemma 2.4. There exists an absolute constant $M_3 > 0$ such that the following holds. If*

- *G is a graph with $m \geq M_3$ edges,*
- *$f(G) = m/2 + \delta$,*
- *$\sqrt{m/8} + cm^{1/4} \leq \delta \leq \sqrt{m/8} + Cm^{1/4}$,*
- *$V(G) = V_1 \cup V_2$ is a partition such that $e(v, V_1) \leq e(v, V_2)$ for every $v \in V_1$, and $e(V_1) \geq m/4 + \sqrt{m/32} - dm^{1/4}$*

then there exists $v \in V_1$, such that

$$e(v, V_1) \leq \sqrt{m/2} + \frac{c}{6}\sqrt{\delta} \quad \text{and} \quad e(v, V_2) \leq \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(v, V_1).$$

Proof. Let G be a graph with m edges and assume that $f(G) = m/2 + \delta$, with $\sqrt{m/8} + cm^{1/4} \leq \delta \leq \sqrt{m/8} + Cm^{1/4}$. Let $V(G) = V_1 \cup V_2$ be a partition such that $e(v, V_1) \leq e(v, V_2)$ for every $v \in V_1$, and $e(V_1) \geq m/4 + \sqrt{m/32} - dm^{1/4}$. Let

$$T = \left\{v \in V_1 : e(v, V_1) > \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right\}$$

and

$$S = \left\{v \in V_1 : e(v, V_2) > \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(v, V_1)\right\}.$$

Since $e(v, V_1) \leq e(v, V_2)$ for all $v \in V_1$, we have

$$e(V_1) \leq \frac{1}{2}e(V_1, V_2) \leq \frac{f(G)}{2} = \frac{m}{4} + \frac{\delta}{2}.$$

On the other hand,

$$2e(V_1) \geq \sum_{v \in T} e(v, V_1) > \left(\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right)|T|.$$

Thus

$$|T| < \frac{2e(V_1)}{\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}} \leq \frac{m/2 + \delta}{\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}}.$$

Since $\sqrt{m/8} + cm^{1/4} \leq \delta \leq \sqrt{m/8} + Cm^{1/4}$, there exists an absolute constant $L_1 > 0$ such that when $m \geq L_1$,

$$|T| < \frac{m/2 + \delta}{\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}} < \sqrt{m/2} - \frac{c}{12}\sqrt{\delta}.$$

Hence

$$\sum_{v \in T} e(v, V_1) \leq e(V_1) + e(T) < e(V_1) + \frac{1}{2}\left(\sqrt{m/2} - \frac{c}{12}\sqrt{\delta}\right)^2.$$

So there exists an absolute constant $L_2 > 0$ such that when $m \geq L_2$,

$$\sum_{v \in T} e(v, V_1) \leq e(V_1) + \frac{m}{4} - \frac{c}{24}\sqrt{m\delta/2}.$$

Since $e(V_1, V_2) = \sum_{v \in S} e(v, V_2) + \sum_{v \in V_1 - S} e(v, V_2)$,

$$e(V_1, V_2) \geq \left(1 + \frac{c}{4}\sqrt{\delta/m}\right)\sum_{v \in S} e(v, V_1) + \sum_{v \in V_1 - S} e(v, V_1) = 2e(V_1) + \frac{c}{4}\sqrt{\delta/m}\sum_{v \in S} e(v, V_1).$$

Therefore

$$\sum_{v \in S} e(v, V_1) \leq \frac{4}{c}\sqrt{m/\delta}(e(V_1, V_2) - 2e(V_1)).$$

Since $e(V_1) \geq m/4 + \sqrt{m/32} - dm^{1/4}$ and $e(V_1, V_2) \leq m/2 + \delta$,

$$\begin{aligned}
\sum_{v \in S} e(v, V_1) &\leq \frac{4}{c}\sqrt{m/\delta}\left(\frac{m}{2} + \delta - 2\left(\frac{m}{4} + \sqrt{m/32} - dm^{1/4}\right)\right) \\
&= \frac{4}{c}\sqrt{m/\delta}(\delta - \sqrt{m/8} + 2dm^{1/4}) \\
&\leq \frac{4}{c}(C + 2d)\sqrt{m/\delta}m^{1/4} \quad \text{(because } \delta \leq \sqrt{m/8} + Cm^{1/4}\text{)}.
\end{aligned}$$

Again, because $e(V_1) \geq m/4 + \sqrt{m/32} - dm^{1/4}$,

$$\sum_{v \in V_1} e(v, V_1) = 2e(V_1) \geq e(V_1) + m/4 + \sqrt{m/32} - dm^{1/4}.$$

Since $\delta = \Theta(\sqrt{m})$, there exists an absolute constant $L_3 > 0$ such that when $m \geq L_3$,

$$\sqrt{m/32} - dm^{1/4} > \frac{4}{c}(C + 2d)\sqrt{m/\delta}m^{1/4} - \frac{c}{24}\sqrt{m\delta/2}.$$

Let $M_3 = \max\{L_1, L_2, L_3\}$. Then when $m \geq M_3$,

$$\sum_{v \in V_1} e(v, V_1) > \sum_{v \in S \cup T} e(v, V_1),$$

which implies that $V_1 - (S \cup T) \neq \emptyset$. ∎

Proof of Theorem 1.3. Let c, M be the absolute constants in Theorem 1.1, and let M_1, M_2, M_3 be the absolute constants in Lemmas 2.2, 2.4 and 2.5. Let $d = c/4$, $M' := \max\{M_1, M_2, M_3\}$, $N' := \lceil (2M')^{1/2} \rceil$, and $N = \max\{N', N'', M\}$, where N'' is an absolute constant to be determined later.

Let $m = n^2/2$, where n is an even number and $n \geq N$; so $n \geq M$ and $m \geq M_i$ for $i = 1, 2, 3$, and hence we may apply Theorem 1.1 and Lemmas 2.2 and 2.4 and 2.5.

Let G be a graph with m edges and $f(G) = m/2 + \delta$. By Theorem 1.1, Lemma 2.2 and 2.4, we may assume that

$$\sqrt{m/8} + cm^{1/4} \leq \delta \leq \sqrt{m/8} + Cm^{1/4}.$$

Let $V(G) = U_1 \cup U_2$, with $e(U_1, U_2) = f(G) = \frac{m}{2} + \delta$ and $e(U_1) \geq e(U_2)$. Then $e(v, U_1) \leq e(v, U_2)$ for every $v \in U_1$; otherwise, $e(U_1 - \{v\}, U_2 \cup \{v\}) > e(U_1, U_2) = f(G)$, a contradiction.

We now describe a process similar to that in the proof of Lemma 2.4, to obtain the desired partition.

- Set $V_1^0 := U_1, V_2^0 := U_2$, and let $V(G) = V_1^i \cup V_2^i$ be the partition obtained after i iterations.
- If $e(V_1^i) \leq m/4 + \sqrt{m/32} - dm^{1/4}$, then set $V_1 := V_1^i$ and $V_2 := V_2^i$, and stop.
- If $e(V_1^i) > m/4 + \sqrt{m/32} - dm^{1/4}$ then by Lemma 2.5, there exists a vertex $u_i \in V_1^i$ such that

  $$e(u_i, V_1^i) \leq \sqrt{m/2} + \frac{c}{6}\sqrt{\delta} \quad \text{and} \quad e(u_i, V_2^i) \leq \left(1 + \frac{c}{4}\sqrt{\delta/m}\right) e(u_i, V_1^i).$$

  Set $V_1^{i+1} := V_1^i - \{u_i\}, V_2^{i+1} := V_2^i \cup \{u_i\}$, and repeat the above steps for V_1^{i+1}, V_2^{i+1}.

Note that, in each iteration of the above procedure, we always have $e(v, V_1^i) \leq e(v, V_2^i)$ for every $v \in V_1^i$; thus Lemma 2.5 may be applied for V_1^i, V_2^i. Let $V_1 = V_1^k, V_2 = V_2^k$ be the final partition, obtained after k steps. Then

$$e(V_1) \leq m/4 + \sqrt{m/32} - dm^{1/4},$$

and

$$e(V_1) > m/4 + \sqrt{m/32} - dm^{1/4} - \left(\sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right).$$

Since $e(v, U_1) \leq e(v, U_2)$ for every $v \in U_1$,

$$e(U_1) \leq \frac{1}{2} e(U_1, U_2) = \frac{m}{4} + \frac{\delta}{2}.$$

Since $e(V_1) = e(U_1) - \sum_{i=0}^{k-1} e(u_i, V_1^i)$, we have

$$\sum_{i=0}^{k-1} e(u_i, V_1^i) = e(U_1) - e(V_1) < e(U_1) - m/4 - \sqrt{m/32} + dm^{1/4} + \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}.$$

Then, since $e(V_2) = e(U_2) + \sum_{i=0}^{k-1} e(u_i, V_2^i)$, $e(u_i, V_2^i) \le \left(1 + \frac{c}{4}\sqrt{\frac{\delta}{m}}\right) e(u_i, V_1^i)$ and $e(U_1) \le m/4 + \delta/2$, we have

$$\begin{aligned}
e(V_2) &\le \frac{m}{2} - \delta - e(U_1) + \left(1 + \frac{c}{4}\sqrt{\frac{\delta}{m}}\right) \sum_{i=0}^{k-1} e(u_i, V_1^i) \\
&< \frac{m}{2} - \delta - e(U_1) + \left(1 + \frac{c}{4}\sqrt{\delta/m}\right)\left(e(U_1) - \frac{m}{4} - \sqrt{m/32} + dm^{1/4} + \sqrt{m/2} + \frac{c}{6}\sqrt{\delta}\right) \\
&= \frac{m}{4} - \delta + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta} + \frac{c}{4}\sqrt{\delta/m}\left(e(U_1) - \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta}\right) \\
&\le \frac{m}{4} - \delta + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta} + \frac{c}{4}\sqrt{\delta/m}\left(\frac{\delta}{2} + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta}\right).
\end{aligned}$$

Since $\sqrt{m/8} + cm^{1/4} \le \delta \le \sqrt{m/8} + Cm^{1/4}$, there exists an absolute constant $N_0 > 0$ such that $dm^{1/4} + (c/6)\sqrt{\delta} < \sqrt{m/32}$ when $n \ge N_0$. Hence,

$$\begin{aligned}
e(V_2) &< \frac{m}{4} - \delta + 3\sqrt{m/32} + dm^{1/4} + \frac{c}{6}\sqrt{\delta} + \frac{c}{4}\sqrt{\delta/m}\left(\frac{\delta}{2} + 4\sqrt{\frac{m}{32}}\right) \\
&= \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} - \delta + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)\sqrt{\delta} + \frac{c}{8\sqrt{m}}\delta^{3/2}.
\end{aligned}$$

We now consider

$$h(\delta) := -\delta + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)\sqrt{\delta} + \frac{c}{8\sqrt{m}}\delta^{3/2}.$$

Differentiating $h(\delta)$ with respect to δ, we get

$$h'(\delta) = -1 + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)\frac{1}{2\sqrt{\delta}} + \frac{3c}{16\sqrt{m}}\sqrt{\delta}.$$

Since $\sqrt{m/8} + cm^{1/4} \le \delta \le \sqrt{m/8} + Cm^{1/4}$, there exists an absolute constant N_1 such that whenever $n \ge N_1$, $h'(\delta) < 0$ and $\sqrt{m/8} + cm^{1/4} < \sqrt{m}$. Thus, we take $N'' := \max\{N_1, N_0\}$ in defining N. So $h(\delta)$ is a decreasing function when $n \ge N$. Hence, for $n \ge N$,

$$\begin{aligned}
h(\delta) &\le h(\sqrt{m/8} + cm^{1/4}) \\
&\le -(\sqrt{m/8} + cm^{1/4}) + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)\sqrt{\sqrt{m/8} + cm^{1/4}} + \frac{c}{8\sqrt{m}}\left(\sqrt{m/8} + cm^{1/4}\right)^{3/2} \\
&< -(\sqrt{m/8} + cm^{1/4}) + \left(\frac{c}{6} + \frac{c}{\sqrt{32}}\right)m^{1/4} + \frac{1}{8}cm^{1/4} \quad (\text{since } \sqrt{m/8} + cm^{1/4} < \sqrt{m}) \\
&= -\sqrt{m/8} + \left(-1 + \frac{1}{6} + \frac{1}{\sqrt{32}} + \frac{1}{8}\right)cm^{1/4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
e(V_2) &\leq \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} + h(\delta) \\
&\leq \frac{m}{4} + 3\sqrt{m/32} + dm^{1/4} - \sqrt{m/8} + \left(-1 + \frac{1}{6} + \frac{1}{\sqrt{32}} + \frac{1}{8}\right) cm^{1/4} \\
&< \frac{m}{4} + \sqrt{m/32} - dm^{1/4} \quad \text{(since } d = c/4\text{)}.
\end{aligned}$$

This completes the proof of Theorem 1.3.

3 k-Partitions

In this section we generalize the proof of Alon in [1] to prove Theorem 1.4. We need the following lemma which appears in several articles, for example, as Lemma 2.1 in [1].

Lemma 3.1 *Let $G = (V, E)$ be an s-colorable graph with m edges. Then for any positive integer $k \leq s$,*

$$f_k(G) \geq \frac{t(s,k)}{\binom{s}{2}} m, \quad \textit{where} \quad t(s,k) = \sum_{1 \leq i < j \leq k} \left\lfloor \frac{s+i-1}{k} \right\rfloor \left\lfloor \frac{s+j-1}{k} \right\rfloor.$$

We will be using Lemma 3.1 for $t(ks, k)$. Note that

$$t(ks,k) = s^2 \binom{k}{2} \text{ and } t(ks,k)/\binom{ks}{2} = \frac{k-1}{k} + \frac{k-1}{k}\frac{1}{ks-1}.$$

Another result we need is due to Bollobás and Scott [7].

Lemma 3.2 (*Bollobás and Scott*) *Let $k \geq 2$ be an integer and let G be a graph with m edges. Then there exists a partition $V(G) = V_1 \cup \ldots \cup V_k$ such that*

$$e(V_1, ..., V_k) \geq \frac{k-1}{k} m + \frac{k-1}{2k}(\sqrt{2m + 1/4} - 1/2) + O(k),$$

where the $O(k)$ term is $(-k^2 + 4k - 4)/(8k)$.

Proof of Theorem 1.4. Fix $k \geq 2$, and let $\epsilon = \frac{1}{16\sqrt{k}}$, $c(k) = \frac{2^{1/4}}{8}\epsilon$, and $N(k) = 32^2 k^3$. (We do not attempt to optimize these constants.) Let n be an even integer such that $n \geq N(k)$. Consider an arbitrary graph G with $m = n^2/2$ edges. Let s denote the unique integer satisfying $n - \epsilon\sqrt{n} + 1 < ks \leq n - \epsilon\sqrt{n} + k + 1$.

Claim 1. We may assume that $\chi(G) \geq ks + 1$.

For, suppose G is ks-colorable. Then

$$\begin{aligned}
f_k(G) &\geq \frac{t(ks,k)}{\binom{ks}{2}} m \quad \text{(by Lemma 3.1)}\\
&= \frac{k-1}{k} m + \frac{k-1}{k} \frac{m}{ks-1}\\
&\geq \frac{k-1}{k} m + \frac{k-1}{2k} \frac{n^2}{n-\epsilon\sqrt{n}+k} \quad \text{(since } m = n^2/2 \text{ and } ks \leq n-\epsilon\sqrt{n}+k+1)\\
&\geq \frac{k-1}{k} m + \frac{k-1}{2k}\left(n + \frac{\epsilon}{2}\sqrt{n}\right) \quad \text{(since } n \geq N(k) = 32^2k^3 \text{ and } \epsilon = 1/(16\sqrt{k}))\\
&\geq \frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + \frac{1}{8}\epsilon(2m)^{1/4} \quad \text{(since } m = n^2/2 \text{ and } (k-1)/(4k) \geq 1/8).
\end{aligned}$$

This proves Claim 1.

Let $H \subseteq G$ such that $\chi(H) = \chi(G)$ and H is vertex-critical. Then

$$\delta(H) \geq \chi(G) - 1 \geq ks \geq n - \epsilon\sqrt{n}.$$

Since $e(H) \leq n^2/2$, $|V(H)| \leq 2e(H)/\delta(H) \leq n + 2\epsilon\sqrt{n}$. Then there are at least $n - 4\epsilon\sqrt{n}$ color classes of size 1 in any proper coloring of H using $\chi(G)$ colors. So there exists a complete subgraph R of G with $|V(R)| := n - r \geq n - 4\epsilon\sqrt{n}$.

Note that $r \leq 4\epsilon\sqrt{n}$ and

$$e(R) = \binom{n-r}{2} = \frac{n^2}{2} - (2r+1)\frac{n}{2} + \frac{r(r+1)}{2}.$$

Hence,

$$\sqrt{e(R)} = \frac{n-r+1/2}{\sqrt{2}} + O(1) \geq \sqrt{m} - \epsilon\sqrt{8n} + O(1),$$

where the $O(1)$ term is $1/(2\sqrt{2}) - 1/8$. Let $W := V(G) - V(R)$. Then

$$e(W) + e(R,W) = m - e(R) = r(n-r) + \frac{n}{2} + \frac{r^2-r}{2}.$$

Claim 2. We may assume that $e(W) \leq n/(8k)$.

Otherwise, assume $e(W) \geq n/(8k)$. By Lemma 3.2, there exist k-partitions $W = \bigcup_{i=1}^k W_i$ and $V(R) = \bigcup_{i=1}^k R_i$ such that

$$\begin{aligned}
e(W_1,...,W_k) &\geq \frac{k-1}{k} e(W) + \frac{k-1}{2k}\sqrt{2e(W)} + O(k)\\
&\geq \frac{k-1}{k} e(W) + \frac{k-1}{4k}\sqrt{n/k} + O(k),
\end{aligned}$$

and

$$\begin{aligned}
e(R_1,...,R_k) &\geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k)\\
&\geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k}4\epsilon\sqrt{n} + O(k).
\end{aligned}$$

For any permutation $\pi \in [k]$, we define $e(\pi) := \sum_{i=1}^{k} e(W_i, R_{\pi(i)})$; then

$$\sum_{\pi\in[k]} e(\pi) = (k-1)!\ e(W, R).$$

Thus there exists a permutation $\pi \in [k]$ such that $e(\pi) \leq (k-1)!\ e(W,R)/k! = e(W,R)/k$. So

$$\begin{aligned}
&e(W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)}) \\
=&e(W,R) - e(\pi) + e(W_1, ..., W_k) + e(R_1, ..., R_k) \\
\geq&\frac{k-1}{k}e(W,R) + \frac{k-1}{k}e(W) + \frac{k-1}{4k}\sqrt{n/k} + \frac{k-1}{k}e(R) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k}4\epsilon\sqrt{n} + O(k) \\
=&\frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + \frac{k-1}{k}(2m)^{1/4}\left(\frac{1}{4\sqrt{k}} - \frac{2}{16\sqrt{k}}\right) + O(k) \\
\geq&\frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + \frac{1}{8\sqrt{k}}(2m)^{1/4} + O(k).
\end{aligned}$$

Note that the $O(k)$ term here is $2(-k^2+4k-4)/(8k) + 1/(2\sqrt{2}) - 1/8$. Since $m > n^2/2 > (32^2k^3)^2/2$, we see that

$$e(W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)}) \geq \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m} + c(k)m^{1/4}.$$

So the assertion of the theorem holds with the partition $W_1 \cup R_{\pi(1)}, ..., W_k \cup R_{\pi(k)}$, completing the proof of Claim 2.

Let $v_1, v_2, ..., v_{n-r}$ be the vertices of R and let $d_i := |N(v_i) \cap W|$ for $1 \leq i \leq n-r$, where $d_1 \leq d_2 \leq ... \leq d_{n-r}$. Since R is complete, a balanced k-partition of $V(R)$ (i.e., the sizes of parts differ by at most 1) gives $f_k(R)$. So let $V(R) = \bigcup_{i=1}^{k} R_i$ be a k-partition such that $\lfloor \frac{n-r}{k} \rfloor = |R_1| \leq |R_2| \leq ... \leq |R_k| = \lceil \frac{n-r}{k} \rceil$, and let $R_1 = \{v_1, ..., v_{\lfloor \frac{n-r}{k} \rfloor}\}$. Then

$$e(R_1, ..., R_k) \geq \frac{k-1}{k}e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k),$$

where the $O(k)$ term is $(-k^2+4k-4)/(8k)$. Let

$$D := \frac{\sum_{1\leq i\leq n-r} d_i}{n-r} = \frac{e(W,R)}{n-r}, \text{ and } D_0 := D - \lfloor D \rfloor.$$

Then, since $e(W) + e(R, W) = r(n-r) + n/2 + (r^2-r)/2$, we have

$$D_0 = \frac{n/2 + (r^2-r)/2 - e(W)}{n-r}.$$

Since $r \leq 4\epsilon\sqrt{n}, n \geq N(k)$ and $e(W) \leq n/(8k)$, we conclude that

$$\min\{D_0, 1-D_0\} \geq \frac{n}{5(n-r)}.$$

Hence, D differs from an integer by at least $\frac{n}{5(n-r)}$.

Claim 3. $e(W, R_2 \cup ... \cup R_k) \geq \frac{k-1}{k} e(W,R) + \frac{n}{5k}$.

To see this, let us consider two cases. If $d_{\lfloor \frac{n-r}{k} \rfloor} \geq D$ then $d_{\lfloor \frac{n-r}{k} \rfloor} \geq D + \frac{n}{5(n-r)}$ by integrality; so Claim 3 follows as

$$e(W, R_2 \cup ... \cup R_k) = \sum_{i > \lfloor \frac{n-r}{k} \rfloor} d_i \geq \frac{k-1}{k}(n-r)\left(D + \frac{n}{5(n-r)}\right) = \frac{k-1}{k} e(W,R) + \frac{k-1}{5k} n.$$

Otherwise, $d_{\lfloor \frac{n-r}{k} \rfloor} \leq D$; then $d_{\lfloor \frac{n-r}{k} \rfloor} \leq D - \frac{n}{5(n-r)}$ by integrality. Hence

$$e(W, R_1) = \sum_{i \leq \lfloor \frac{n-r}{k} \rfloor} d_i \leq \frac{n-r}{k}\left(D - \frac{n}{5(n-r)}\right) \leq \frac{1}{k} e(W,R) - \frac{n}{5k};$$

so $e(W, R_2 \cup ... \cup R_k) \geq \frac{k-1}{k} e(W,R) + \frac{n}{5k}$, completing the proof of Claim 3.

Now, consider the k-partition $W \cup R_1, R_2, ..., R_k$ of $V(G)$. Since $e(R_1, \ldots, R_k) \geq \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + O(k)$ and by Claim 3, We have

$$\begin{aligned}
& e(W \cup R_1, R_2, \ldots, R_k) \\
=& e(R_1, R_2, \ldots, R_k) + e(W, R_2 \cup \ldots \cup R_k) \\
\geq& \frac{k-1}{k} e(R) + \frac{k-1}{2k}\sqrt{2e(R)} + \frac{k-1}{k} e(W,R) + \frac{n}{5k} + O(k) \\
\geq& \frac{k-1}{k} m - \frac{k-1}{k} e(W) + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{2k} 4\epsilon\sqrt{n} + \frac{n}{5k} + O(k) \quad (\text{as } \sqrt{e(R)} \geq \sqrt{m} - \epsilon\sqrt{8m} + O(1)) \\
\geq& \frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} - \frac{k-1}{k}\frac{\sqrt{2m}}{8k} - \frac{k-1}{2k} 4\epsilon (2m)^{1/4} + \frac{\sqrt{2m}}{5k} + O(k) \quad (\text{by Claim 2}) \\
\geq& \frac{k-1}{k} m + \frac{k-1}{2k}\sqrt{2m} + c(k) m^{1/4}.
\end{aligned}$$

The last inequality holds, since $n \geq N(k) \geq 32^2 k^3$ and the $O(k)$ term is around $(-k^2 + 4k - 4)/(8k)$. ∎

4 Concluding remarks

In [1], Alon mentioned that the technique used to prove Theorem 1.1 may be applied to improve the lower bound on $f(m)$ for other values of m, and the reason why $m = n^2/2$ is used is to make the proof concise. Therefore, our upper bound on $g(m)$ holds for certain other values of m as well. Alon [1] also indicated that it would be interesting to find the precise value of $f(m)$ for every integer m; the same applies to $g(m)$.

Theorem 1.3 is best possible up to the constant d. To see this, let $m = \frac{n^2+1}{2}$. The lower bound on $f(m)$ remains unchanged since the proof in [1] works for values of m which differ from $n^2/2$ by a constant; so our proof also gives the same upper bound on $g(m)$. Let G be the vertex-disjoint union of K_n and K_k, where n is odd, k is even, and $n = k(k-1) - 1$. Let $m := e(G) = \binom{n}{2} + \binom{k}{2}$. An easy calculation shows that

$$g(G) = \binom{\frac{n+1}{2}}{2} + \binom{\frac{k}{2}}{2} = \frac{m}{4} + \sqrt{m/32} - \frac{(2m)^{1/4}}{4} + O(1).$$

This shows that for $m = \frac{n^2+1}{2}$, $g(m) \geq m/4 + \sqrt{m/32} - (2m)^{1/4}/8 + O(1)$.

In our proof of Theorem 1.3, $d = c/4$, where c is the constant from Theorem 1.1. The calculation in the end of the proof of Theorem 1.3 requires that $d - c + O(1) \leq -d$, where the $O(1)$ term may be made arbitrarily small when m is sufficient large. So one can show that for any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that when $n \geq N$,

$$g(m) \leq \frac{m}{4} + \sqrt{m/32} - \left(\frac{c}{2} - \epsilon\right) m^{1/4}.$$

We do not know if one can get rid of the ϵ.

Our proofs of Lemmas 2.3 and 2.4 may be modified to show that if G is graph with m edges, $f(G) = m/2 + \delta$, and $\delta \leq \alpha m$, then for sufficiently large m, there is a partition $V(G) = V_1 \cup V_2$ such that

$$\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} - \frac{\delta}{2} + \sqrt{m/8} + \beta\sqrt{\delta} + \gamma\frac{\delta^{3/2}}{\sqrt{m}},$$

where α is an absolute constant and β, γ are constants depend on α only. When $\delta = O(m^t)$, where $t < \frac{2}{3}$, this bound is better than Lemma 2.1, since $\sqrt{m/8}$ dominates $\sqrt{\delta}$, $\delta^{3/2}/\sqrt{m}$ (while $3\sqrt{m}$ dominates $10\delta^2/m$ in Lemma 2.1).

Another conclusion we may draw from the proofs of Lemmas 2.3 and 2.4 is that: If $f(G) \geq m/2 + \sqrt{m/8} + \alpha$, where $\alpha = \Theta(m^t)$ and $\frac{1}{4} < t < \frac{1}{2}$, then $g(G) \leq m/4 + \sqrt{m/32} + O(m^{1/4}) - \alpha/2$.

We conclude our discussion with the following natural question: Is it also true that the limsup of

$$\frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m + 1/4} - 1/2) - g_k(m)$$

tends to infinity when m tends to infinity? Theorem 1.4 and results of Bollobás and Scott in [9] seem to be relevant.

References

[1] N. Alon, Bipartite subgraphs, *Combinatorica* **16** (1996) 301–311.

[2] N. Alon, B. Bollobás, M. Krivelevich and B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, *J. Combin. Theory Ser. B* **88** (2003) 329–346.

[3] P. Berman and M. Karpinski, On some tighter inapproximability results, (extended abstract) Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, **1644** (1999) 200–209.

[4] B. Bollobás and A. D. Scott, On judicious partitions, *Period. Math. Hungar.* **26** (1993) 127–139.

[5] B. Bollobás and A. D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* **19** (1999) 473–486.

[6] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Structures and Algorithms* **21** (2002) 414-430.

[7] B. Bollobás and A. D. Scott, Better bounds for Max Cut, in Contemporary Comb, Bolyai Soc Math Stud 10, *János Bolyai Math Soc*, Budapest, (2002) 185–246.

[8] B. Bollobás and A. D. Scott, Judicious partitions of bounded-degree graphs, *J. Graph Theory* **46** (2004) 131–143.

[9] B. Bollobás and A. D. Scott, Max k-cut and judicious k-partitions, *Discrete Math.* **310** (2010) 2126–2139.

[10] C. S. Edwards, Some extremal properties of bipartite graphs, *Canadian J. math.* **25** (1973) 475–485.

[11] C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, in *Proc. 2nd Czechoslovak Symposium on Graph Theory*, Prague (1975) 167–181.

[12] P. Erdős, Some recent problems in Combinatorics and Graph Theory, Proc. 26 th Southeastern International Conference on Graph Theory, Combinatorics and Computing, Boca Raton, 1995, Congressus Numerantium.

[13] U. Feige, M. Karpinski, and M. Langberg, Improved approximation of max-cut on graphs of bounded degree, *J. Algorithms* **43** (2002) 201–219.

[14] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability using semidefinite programming, *J. ACM* **42** (1995) 1115–1145.

[15] R. M. Karp, Reducibility among combinatorial problems, in *Complexity of Computer Computations*, (R. Miller and J. Thatcher, eds) Plenum Press, New York, (1972) 85–103.

[16] A. Scott, Judicious partitions and related problems, in *Surveys in Combinatorics* (2005) pp. 95–117, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.

[17] F. Shahrokhi and L. A. Székely, The complexity of the bottleneck graph bipartition problem, *J. Combin. Math. Combin. Comput.* **15** (1994) 221–226.

[18] B. Xu and X. Yu, Better bounds for k-partitions of graphs, *Combinatorics, Probability and Computing* **20** (2011) 631-640.

[19] M. Yannakakis, Node- and Edge-Deletion NP-Complete Problems, *STOC* (1978) 253–264.